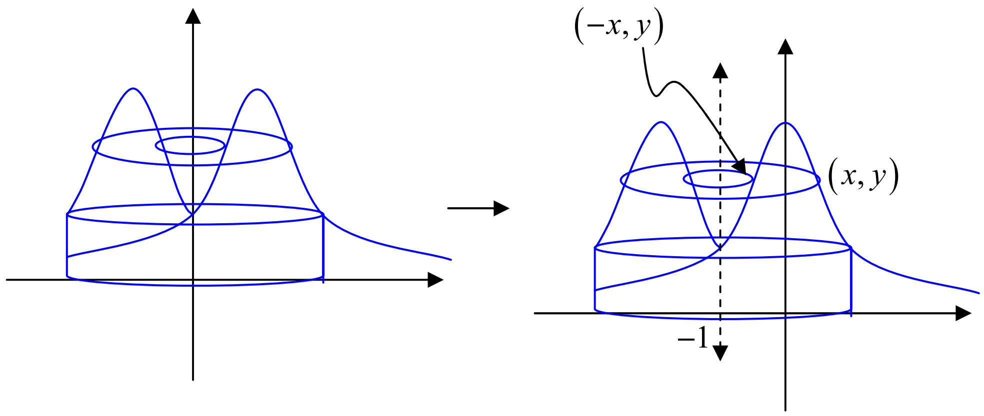
Practice Questions

Volumes

(a) (i) By slicing method: The solid consists of two parts: The upper part V_1 (for $y \geq \frac{1}{2}$) and the lower part V_2 (for $0 \leq y \leq \frac{1}{2}$).



For the upper part, it might be easier to shift the curve so that it is rotated about the line $x = -1$ and its new equation is $y = \dfrac{1}{x^2+1}$.

Consider a cross section, which is a ring, of radii $x+1$ and $-x+1$.

The area of the ring $= \pi\left((1+x)^2-(1-x)^2\right) = \pi\left(\left(1+2x+x^2\right)-\left(1-2x+x^2\right)\right) = 4\pi x$

The volume of the slice $= \partial V_1 = 4\pi x \partial y$.

$\therefore V_1 = 4\pi \int_{\frac{1}{2}}^{1} x\, dy.$

Let's change dy into dx : $y = \dfrac{1}{1+x^2}, \therefore dy = \dfrac{-2x}{(1+x^2)^2} dx.$

When $y=\frac{1}{2}, x=1$; When $y=1, x=0$.

$\therefore V_1=4\pi\int_1^0 \frac{-2x^2}{(1+x^2)^2}dx=8\pi\int_0^1 \frac{x^2}{(1+x^2)^2}dx.$

Let $x=\tan\theta, dx=\sec^2\theta\, d\theta$.

When $x=0, \theta=0$; When $x=1, \theta=\frac{\pi}{4}$.

$$\therefore V_1=8\pi\int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{\sec^4\theta}\sec^2\theta\, d\theta=8\pi\int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{\sec^2\theta}d\theta=8\pi\int_0^{\frac{\pi}{4}} \sin^2\theta\, d\theta$$

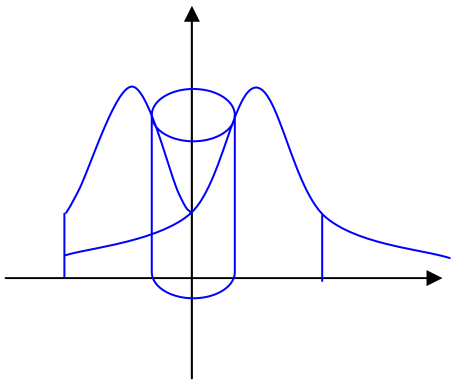
$$=4\pi\int_0^{\frac{\pi}{4}}(1-\cos 2\theta)\, d\theta=4\pi\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}=4\pi\left(\frac{\pi}{4}-\frac{1}{2}\right)$$

$$=\pi^2-2\pi$$

The lower part is a cylinder, with radius = 2, height = $\frac{1}{2}$, $\therefore V_2=\pi.2^2.\frac{1}{2}=2\pi$.

$\therefore$ The total volume = $\pi^2-2\pi+2\pi=\pi^2$ units^3..

(ii) By the shells method: Consider a hollow shell of radii x and $x+\partial x$, height y.



The volume of the shell $\partial V=\pi\left((x+\partial x)^2-x^2\right)y \simeq 2\pi xy\partial x$.

$$V=2\pi\int_0^2 xy\, dx=2\pi\int_0^2 \frac{x}{(x-1)^2+1}dx$$

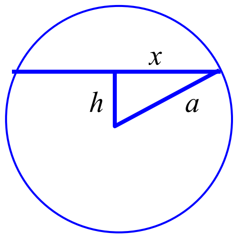
$$=2\pi\int_0^2 \frac{x-1}{(x-1)^2+1}dx+2\pi\int_0^2 \frac{1}{(x-1)^2+1}dx$$

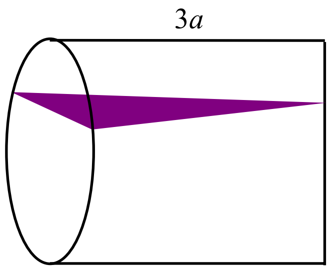
$$=\pi\left[\ln\left((x-1)^2+1\right)\right]_0^2+2\pi\left[\tan^{-1}(x-1)\right]_0^2$$

$$=\pi\left(\ln 2-\ln 2\right)+2\pi\left(\frac{\pi}{4}+\frac{\pi}{4}\right)=\pi^2 \text{ units}^3.$$

(b) (i) The slice is a triangular prism, of base $2x$, height $3a$, and thickness ∂h .

The area of the slice $= \frac{1}{2}.2x.3a = 3ax$, where $x = \sqrt{a^2 - h^2}$.



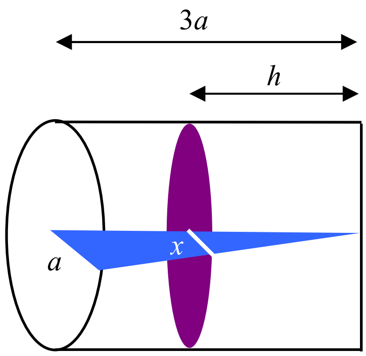


The volume of the slice $\partial V = 3ax.\partial h = 3a\sqrt{a^2 - h^2}.\partial h$

$\therefore V = 3a \times 2\int_0^a \sqrt{a^2 - h^2}\, dh = 6a \times \frac{1}{4}$ the area of a circle of radius a

$= 6a \times \frac{1}{4}\pi a^2 = \frac{3\pi a^3}{2}$ units^3.

(ii) The slice is an elliptical prism, whose thickness = ∂h , semi-major axis = a, and semi-minor axis = x, where $\frac{x}{a} = \frac{h}{3a}$, by similar triangles, $\therefore x = \frac{h}{3}$.



The area of the slice is $\frac{\pi a h}{3}$.

The volume of the slice $\partial V = \frac{\pi a h}{3}.\partial h$

$\therefore V = \frac{\pi a}{3}\int_0^{3a} h\, dh = \frac{\pi a}{3}\left[\frac{h^2}{2}\right]_0^{3a} = \frac{\pi a}{3}\frac{9a^2}{2} = \frac{3\pi a^3}{2}$ units^3.