Practice Questions

Solutions

Polynomials

(a) Let the roots be $\alpha, -\alpha, \beta$ and γ.

$\sum \alpha = \beta + \gamma = -2$ (1)

$\sum \alpha\beta\gamma = -\alpha^2\beta - \alpha^2\gamma = 4$ (2)

$\prod \alpha = -\alpha^2\beta\gamma = -4$ (3)

(2) can be rewritten as $-\alpha^2(\beta+\gamma) = 4$ (4)

$\frac{(4)}{(1)}$ gives $\frac{-\alpha^2(\beta+\gamma)}{\beta+\gamma} = \frac{4}{-2}, \therefore \alpha^2 = 2$

Put $\alpha^2 = 2$ to (3) gives $\beta\gamma = 2.$ (5)

$\therefore$ From (1) and (5), β and γ are the roots of the eqn $x^2 + 2x + 2 = 0$

$\therefore x^4 + 2x^3 - 4x - 4 = (x^2 + 2x + 2)(x^2 - 2)$, by inspection.

$\therefore$ The roots are $x = -1 \pm i$ and $\pm\sqrt{2}$.

(b) (i) Let the eqn be $x^3 + bx^2 + cx + d = 0$, where b, c and d are rational numbers.

$\alpha + \beta + \gamma = \frac{9}{2} = -b, \therefore b = -\frac{9}{2}$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2\sum \alpha\beta$

$\frac{25}{4} = \left(\frac{9}{2}\right)^2 - 2\sum \alpha\beta, \therefore \sum \alpha\beta = 7, \therefore c = 7.$

$\alpha^3+\beta^3+\gamma^3=-b\sum\alpha^2-c\sum\alpha-3d$

$$\frac{33}{8}=\left(\frac{9}{2}\right)\left(\frac{25}{4}\right)-7\left(\frac{9}{2}\right)-3d,\therefore d=-\frac{5}{2}$$

The eqn is $x^3-\frac{9}{2}x^2+7x-\frac{5}{2}=0$, or on multiplying by 2, $2x^3-9x^2+14x-5=0$

(ii) Let $P(x)=2x^3-9x^2+14x-5$.

By trial and error, $P\left(\frac{1}{2}\right)=\frac{1}{4}-\frac{9}{4}+7-5=0,\therefore\frac{1}{2}$ is a root, or $2x-1$ is a factor.

$2x^3-9x^2+14x-5=(2x-1)(x^2-4x+5)$, by inspection.

$\therefore$ The roots are $x=\frac{1}{2}$ and $2\pm i$.

(c) (i) $-ki$ is also a root because the coefficients are real.
(ii) Substitute x by ki,

$-k^3i-ak^2+bki+c=0$

$(-ak^2+c)+i(-k^3+bk)=0$

$\therefore -ak^2+c=0$ and $k(-k^2+b)=0$, by equating the real parts and the imaginary parts.

But $k\neq 0,\therefore k^2=b$.

(iii) Put $k^2=b$ to $-ak^2+c=0$ gives $-ab+c=0$, or $ab=c$.

(iv) If 2 is a root of the equation then we can find a, $\sum\alpha=2+ki-ki=-a,\therefore a=-2$.
From $ab=c$, we can tell that $c=-2b$, so there is an infinite set of solutions for b and c so long it satisfies $c=-2b$. For example, $x^3-2x^2+x-2=0$, where $b=1$, $c=-2$, has three roots 2 and $\pm i$.

(d) $1=\text{cis}\,0=\text{cis}\,2k\pi,\therefore\sqrt[3]{1}=\text{cis}\frac{2k\pi}{3}$, where $k=0,\pm1$ and $\sqrt[9]{1}=\text{cis}\frac{2k\pi}{9}$, where $k=0,\pm1,\pm2,$ ±3 and ±4. Of the nine ninth roots of 1, the roots which correspond to $k=0$, 3 and -3 also are cube roots of 1.

But $z^9-1=\left(z^3-1\right)\left(z^6+z^3+1\right)$.

$\therefore$ The solutions of $z^6+z^3+1=0$ are $x=\text{cis}\frac{2k\pi}{9},k=\pm1,\pm2,\pm4$.

$$\therefore\sum\alpha=\text{cis}\frac{2\pi}{9}+\text{cis}\left(-\frac{2\pi}{9}\right)+\text{cis}\frac{4\pi}{9}+\text{cis}\left(-\frac{4\pi}{9}\right)+\text{cis}\frac{8\pi}{9}+\text{cis}\left(-\frac{8\pi}{9}\right)$$

$$=2\left(\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{8\pi}{9}\right)=-\frac{b}{a}=0$$

$\therefore\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}-\cos\frac{\pi}{9}=0$, since $\cos\frac{8\pi}{9}=-\cos\frac{\pi}{9}$.

$\therefore\cos\frac{\pi}{9}-\cos\frac{2\pi}{9}-\cos\frac{4\pi}{9}=0.$