Practice Questions

Mechanics

(a) (i) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2}\left((4-x)^2 - 2x(4-x)\right) = \frac{1}{2}(4-x)(4-3x)$

(ii) $v^2 = x(4-x)^2 \Rightarrow v = (4-x)\sqrt{x}.$

$\frac{dx}{dt} = (4-x)\sqrt{x}$

$\int \frac{dx}{\sqrt{x}(4-x)} = \int dt$

Let $u^2 = x, \therefore 2u\,du = dx$

$t = \int \frac{dx}{\sqrt{x}(4-x)} = \int \frac{2u\,du}{u\left(4-u^2\right)} = \int \frac{2du}{4-u^2} = \frac{1}{2}\int \left(\frac{1}{2+u} + \frac{1}{2-u}\right)du$

$= \frac{1}{2}\ln\frac{2+u}{2-u} + C = \frac{1}{2}\ln\frac{2+\sqrt{x}}{2-\sqrt{x}} + C$

When $t = 0, v = 0, \therefore C = 0.$

$\therefore t = \frac{1}{2}\ln\frac{2+\sqrt{x}}{2-\sqrt{x}}.$

$e^{2t} = \frac{2+\sqrt{x}}{2-\sqrt{x}}.$

$2e^{2t} - e^{2t}\sqrt{x} = 2 + \sqrt{x}.$

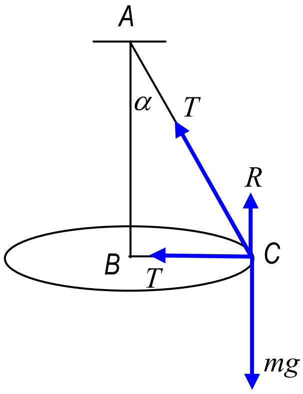
$2e^{2t} - 2 = \sqrt{x}\left(1+e^{2t}\right).$

$\therefore \sqrt{x} = \frac{2e^{2t}-2}{1+e^{2t}} = 2\left(\frac{e^{2t}-1}{1+e^{2t}}\right), \therefore x = 4\left(\frac{e^{2t}-1}{e^{2t}+1}\right)^2$ m.

(iii) At $x = \frac{4}{3}$ m its acceleration is zero, the particle starts slowing down. The time at the moment is $t=\frac{1}{2}\ln\frac{2+\sqrt{\frac{4}{3}}}{2-\sqrt{\frac{4}{3}}}=\frac{1}{2}\ln\frac{\sqrt{3}+1}{\sqrt{3}-1}=\frac{1}{2}\ln\frac{\left(\sqrt{3}+1\right)^2}{2}=\frac{1}{2}\ln\left(2+\sqrt{3}\right)$ seconds.

(iv) The next time the particle stops (v = 0) is at x = 4 m. At that moment, its acceleration is also 0, so it stops there permanently, however, this is its limiting position ($t\to\infty$).

(b)



(i) $BC=r, AC=3.6-r$

$(3.6-r)^2=2.4^2+r^2$

$r^2-7.2r+3.6^2=2.4^2+r^2$

$\therefore r=\frac{3.6^2-2.4^2}{7.2}=1. \therefore BC=1$ m.

(ii) $\tan\alpha=\frac{1}{2.4}, \therefore \alpha=22.6°$.

Let T be the tension of the string, R the normal reaction. Resolving the forces,

Vertically, $T\cos\alpha+R=mg$ (1)

Horizontally, $T\sin\alpha+T=mr\omega^2$ (2)

We want to find the maximum value of ω so that the particle is still in touch with the table, i.e. $R>0$.

Combining (1) and (2), $R=mg-T\cos\alpha=mg-\frac{mr\omega^2\cos\alpha}{(1+\sin\alpha)}$.

$R>0$, so $mg-\frac{mr\omega^2\cos\alpha}{(1+\sin\alpha)}>0$.

$\therefore \omega^2<\frac{g(1+\sin\alpha)}{r\cos\alpha}=\frac{10\times(1+\sin 22.6°)}{\cos 22.6°}=15. \therefore \omega<3.9$ rad/s.

$\therefore$ The maximum value of ω is 3.9 rad/s.

(c) (i) Resolving the forces,

Tangentially to the circle, $mr\ddot{\theta}=-F$, where F is the friction. (1)

Radially, $mr\dot{\theta}^2=T$, where T is the tension in the rod. (2)

From (1), $m=1, r=\frac{1}{2}, \therefore \ddot{\theta}=-2F$. (3)

$\frac{d\dot{\theta}}{dt} = -2F, \therefore \int_{\dot{\theta}_0}^{0} d\dot{\theta} = -2F\int_0^5 dt$, where $\dot{\theta}_0$ is the initial angular velocity. (4)

$-\dot{\theta}_0 = -10F, \therefore \dot{\theta}_0 = 10F.$ (5)

Also, (3) can be written as $\frac{\dot{\theta}d\dot{\theta}}{d\theta} = -2F.$

$\int_{\dot{\theta}_0}^{0} \dot{\theta}d\dot{\theta} = -2F\int_0^{2\pi} d\theta$ (6)

$-\frac{1}{2}\dot{\theta}_0^{\ 2} = -4\pi F, \therefore \dot{\theta}_0^{\ 2} = 8\pi F.$

Comparing with (5), $100F^2 = 8\pi F, \therefore F = \frac{8\pi}{100} = \frac{2\pi}{25}$ newtons.

(ii) From (5), the initial angular velocity $\dot{\theta}_0 = 10F = 10 \times \frac{2\pi}{25} = \frac{4\pi}{5}$ rad/s.

(iii) Reintegrate (6), using $F = \frac{2\pi}{25}, \dot{\theta}_0 = \frac{4\pi}{5}$ and different limits,

$\int_{\frac{4\pi}{5}}^{\alpha} \dot{\theta}d\dot{\theta} = -\frac{4\pi}{25}\int_0^{\pi} d\theta$, where α is the angular velocity when $\theta = \pi$.

$\frac{1}{2}\left(\alpha^2 - \frac{16\pi^2}{25}\right) = -\frac{4\pi^2}{25}$

$\therefore \alpha^2 = \frac{16\pi^2}{25} - \frac{8\pi^2}{25} = \frac{8\pi^2}{25}, \therefore a = \frac{2\sqrt{2}\pi}{5}$ rad/s.

Reintegrate (4), using $F = \frac{2\pi}{25}, \dot{\theta}_0 = \frac{4\pi}{5}$ and different limits,

$\int_{\frac{4\pi}{5}}^{\frac{2\sqrt{2}\pi}{5}} d\dot{\theta} = -\frac{4\pi}{25}\int_0^{T} dt$, where T_1 is the time to make a half turn.

$\frac{2\sqrt{2}\pi}{5} - \frac{4\pi}{5} = -\frac{4\pi T_1}{25}.$

$\therefore T_1 = \frac{10-5\sqrt{2}}{2}$ s.

From (2), when it has made a half turn the tension $T = \frac{1}{2}\left(\frac{2\sqrt{2}\pi}{5}\right)^2 = \frac{4\pi^2}{25}$ newtons.