Practice Questions

Integration

(a) $\int \frac{dx}{\sqrt{x(4-x)}} = \int \frac{dx}{\sqrt{4-(x-2)^2}} = \sin^{-1}\frac{x-2}{2} + C$.

(b) $\int \cos^3 x \sin^2 x\, dx = \int \cos x\left(1-\sin^2 x\right)\sin^2 x\, dx = \int\left(\sin^2 x - \sin^4 x\right)\cos x\, dx$

$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$.

(c) Let $t = \tan\frac{x}{2}, dt = \frac{1}{2}\sec^2\frac{x}{2}dx, \therefore dx = \frac{2dt}{\sec^2\frac{x}{2}} = \frac{2dt}{1+t^2}$.

When $x = 0, t = 0$; When $x = \frac{\pi}{4}, t = \tan\frac{\pi}{8} = \sqrt{2}-1$. (*)

$$\int_0^{\frac{\pi}{4}} \frac{\tan x}{1+\cos x-\sin x}dx = \int_0^{\sqrt{2}-1} \frac{\frac{2t}{1-t^2}}{1+\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}}\frac{2dt}{1+t^2}$$

$$= \int_0^{\sqrt{2}-1} \frac{4t}{\left(1-t^2\right)\left(1+t^2+1-t^2-2t\right)}dt$$

$$= \int_0^{\sqrt{2}-1} \frac{2t}{\left(1-t^2\right)\left(1-t\right)}dt = \int_0^{\sqrt{2}-1} \frac{2t}{\left(1+t\right)\left(t-1\right)^2}dt$$

$$= \int_0^{\sqrt{2}-1} \left(\frac{-\frac{1}{2}}{1+t} + \frac{\frac{1}{2}}{t-1} + \frac{1}{(t-1)^2}\right)dt$$

$$=\left[-\frac{1}{2}\ln(1+t)+\frac{1}{2}\ln(t-1)-\frac{1}{t-1}\right]_0^{\sqrt{2}-1}$$

$$=-\frac{1}{2}\ln\sqrt{2}+\frac{1}{2}\ln\frac{\sqrt{2}-2}{-1}-\left(\frac{1}{\sqrt{2}-2}-(-1)\right)$$

$$=\frac{1}{2}\ln\frac{2-\sqrt{2}}{\sqrt{2}}-\left(\frac{\sqrt{2}+2}{-2}+1\right)$$

$$=\frac{1}{2}\ln\left(\sqrt{2}-1\right)+\frac{\sqrt{2}}{2}.$$

(*) From $\tan 2x=\dfrac{2\tan x}{1-\tan^2 x}$, let $x=\dfrac{\pi}{8}$ and $\tan\dfrac{\pi}{8}=t$, we have $1=\dfrac{2t}{1-t^2}$, or $t^2+2t-1=0$.

$\therefore t=\dfrac{-2\pm\sqrt{8}}{2}=-1\pm\sqrt{2}.$

As $\tan\dfrac{\pi}{8}>0, \tan\dfrac{\pi}{8}=-1+\sqrt{2}.$

(d) Let $u=\dfrac{\pi}{2}-x, du=-dx$

When $x=0, u=\dfrac{\pi}{2}$; When $x=\dfrac{\pi}{2}, u=0.$

$$\int_0^{\frac{\pi}{2}}\frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}dx=\int_{\frac{\pi}{2}}^{0}\frac{e^{\sin\left(\frac{\pi}{2}-u\right)}}{e^{\sin\left(\frac{\pi}{2}-u\right)}+e^{\cos\left(\frac{\pi}{2}-u\right)}}(-du)=\int_0^{\frac{\pi}{2}}\frac{e^{\cos u}}{e^{\cos u}+e^{\sin u}}du=\int_0^{\frac{\pi}{2}}\frac{e^{\cos x}}{e^{\sin x}+e^{\cos x}}dx$$

Hence, $\displaystyle\int_0^{\frac{\pi}{2}}\frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{e^{\sin x}+e^{\cos x}}{e^{\sin x}+e^{\cos x}}dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}dx=\frac{\pi}{4}.$

(e) $\displaystyle\int_0^1\frac{x^2+2}{(2x+1)(x^2+1)}dx=\int_0^1\left(\frac{\frac{9}{5}}{2x+1}+\frac{-\frac{2}{5}x+\frac{1}{5}}{x^2+1}\right)dx$

$$=\left[\frac{9}{10}\ln(2x+1)-\frac{1}{5}\ln(x^2+1)+\frac{1}{5}\tan^{-1}x\right]_0^1$$

$$=\frac{9}{10}\ln 3-\frac{1}{5}\ln 2+\frac{\pi}{20}.$$

(f) Let $u=x^n, du=nx^{n-1}dx, dv=\sin x, v=-\cos x$

$$\int_0^{\frac{\pi}{2}}x^n\sin x\,dx=\left[-x^n\cos x\right]_0^{\frac{\pi}{2}}+n\int_0^{\frac{\pi}{2}}x^{n-1}\cos x\,dx=n\int_0^{\frac{\pi}{2}}x^{n-1}\cos x\,dx$$

Let $u=x^{n-1}, du=(n-1)x^{n-2}dx, dv=\cos x, v=\sin x$

$$\therefore n\int_0^{\frac{\pi}{2}} x^{n-1}\cos x\,dx = n\left[x^{n-1}\sin x\right]_0^{\frac{\pi}{2}} - n(n-1)\int_0^{\frac{\pi}{2}} x^{n-2}\sin x\,dx$$

$$\therefore u_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)u_{n-2}.$$

Now, $u_6 = 6\left(\frac{\pi}{2}\right)^5 - 30u_4,$

$$u_4 = 4\left(\frac{\pi}{2}\right)^3 - 12u_2,$$

$u_2 = 2\left(\frac{\pi}{2}\right) - 2u_0,$ where $u_0 = \int_0^{\frac{\pi}{2}} \sin x\,dx = \left[-\cos x\right]_0^{\frac{\pi}{2}} = 1$

$$\therefore u_6 = \frac{3\pi^5}{16} - 30\left[\frac{\pi^3}{2} - 12(\pi - 2)\right] = \frac{3\pi^5}{16} - 15\pi^3 + 360\pi - 720.$$