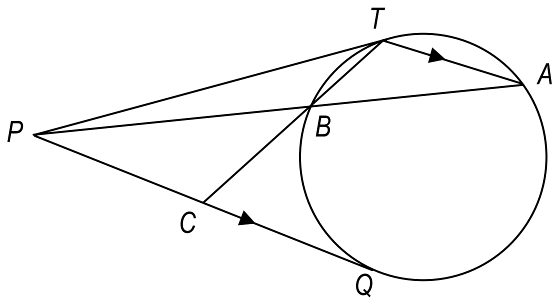
Practice Questions

Harder Extension 1 topics

(a)



(i) In ΔTPC and ΔPBC,
$\angle C$ is common,
$\angle PTC = \angle BPC$ ($\angle PTC = \angle TAP$, angles in alternate segments are equal, and $\angle TAP = \angle BPC$, alternate angles on parallel lines)
$\therefore \Delta TPC \,|||\, \Delta PBC$ (equiangular)

(ii) $\frac{TC}{PC} = \frac{PC}{BC}$ (corresponding sides in similar triangles)
$\therefore PC^2 = TC \times BC$.

(iii) But $CQ^2 = CB.CT$ (intersecting secants theorem)
$\therefore PC^2 = CQ^2, \therefore PC = CQ$, $\therefore C$ is the midpoint of PQ.

(b) The number of words that they are separated by at least two letters = Total – the number of words when they are together – the number of words when they are separated by 1 letter.

Total $= \frac{7!}{2!}$ words.

The two letters S's are together in 6! words.
To separate the two S's by one letter, choose one more letter from the other 5 letters

(5C1) to make a group, so together there are 4 distinct letters and a group, hence, 5! ways, but inside the group of the 3 letters the two *S*'s must not stay in the middle, $\therefore$ only 1 way. $\therefore 5\times 5!$ ways.

$\therefore$ The two letters *S*'s are separated by at least two other letters in $\frac{7!}{2!}-6!-5\times 5! = 1200$ ways.

(c) (i) $f'(x)=\frac{\ln x-1}{(\ln x)^2}$.

For $x\geq 3, \ln 3>1, \therefore f'(x)>0, \therefore f(x)$ is monotonic increasing.

(ii) $\therefore$ For $n\geq 3, \frac{n+1}{\ln(n+1)}>\frac{n}{\ln n}$.

$(n+1)\ln n > n\ln(n+1)$

$\ln n^{n+1} > \ln(n+1)^n$

$\therefore n^{n+1}>(n+1)^n$, since $\ln x$ is increasing for all $x>0$.

(d) (i) $(\cos x+i\sin x)^{2n+1}=\cos(2n+1)x+i\sin(2n+1)x$, by De Moivre's theorem.

$$(\cos x+i\sin x)^{2n+1}=\binom{2n+1}{0}\cos^{2n+1}x+i\binom{2n+1}{1}\cos^{2n}x\sin x-\binom{2n+1}{2}\cos^{2n-1}x\sin^2 x$$
$$-i\binom{2n+1}{3}\cos^{2n-2}x\sin^3 x+\ldots+i(-1)^n\binom{2n+1}{2n+1}\sin^{2n+1}x,$$ by binomial expansion,

noting that the last term contains $i^{2n+1}=(i)^{2n}.(i)=(-1)^n.i$.

Equating the imaginary parts,

$$\sin(2n+1)x=\binom{2n+1}{1}\cos^{2n}x\sin x-\binom{2n+1}{3}\cos^{2n-2}x\sin^3 x+\ldots+(-1)^n\binom{2n+1}{2n+1}\sin^{2n+1}x.$$

$$\therefore \frac{\sin(2n+1)x}{\sin^{2n+1}x}=\binom{2n+1}{1}\cot^{2n}x-\binom{2n+1}{3}\cot^{2n-2}x+\ldots+(-1)^n.$$

(ii) Let $t=\cot^2 x$, the roots of the polynomial $p(t)=\binom{2n+1}{1}t^n-\binom{2n+1}{3}t^{n-1}+\ldots+(-1)^n$ $=0$ are the same as the roots of $\frac{\sin(2n+1)x}{\sin^{2n+1}x}=0$.

Solving $\sin(2n+1)x=0$ gives $x=\frac{k\pi}{2n+1}, k=0,1,2,\ldots,n$. These values of *k* are chosen so that $0\leq x\leq\frac{\pi}{2}$. But when $k=0, \sin x=0$, $\frac{\sin(2n+1)x}{\sin^{2n+1}x}$ is undefined, so $\frac{\sin(2n+1)x}{\sin^{2n+1}x}=0$ only has *n* roots, $x=\frac{k\pi}{2n+1}, k=1,2,\ldots,n$.

$\therefore p(t)$ has exactly *n* roots and they are $t=\cot^2\left(\frac{k\pi}{2n+1}\right), k=1,2,\ldots,n$.

(iii) $\sum_{k=1}^{n}\cot^2\left(\frac{k\pi}{2n+1}\right)=\sum\alpha=\frac{\binom{2n+1}{3}}{\binom{2n+1}{1}}=\frac{\frac{(2n+1)!}{3!(2n-2)!}}{\frac{(2n+1)!}{(2n)!}}=\frac{(2n)(2n-1)}{3!}=\frac{2n(2n-1)}{6}$.

(iv) $\operatorname{cosec}^2 x=\cot^2 x+1, \therefore \sum_{k=1}^{n}\operatorname{cosec}^2\left(\frac{k\pi}{2n+1}\right)=\frac{2n(2n-1)}{6}+n=\frac{4n^2+4n}{6}=\frac{2n(2n+2)}{6}$.

(v) Since $\cot x<\frac{1}{x}<\operatorname{cosec} x, \sum_{k=1}^{n}\cot^2\left(\frac{k\pi}{2n+1}\right)<\sum_{k=1}^{n}\left(\frac{2n+1}{k\pi}\right)^2<\sum_{k=1}^{n}\operatorname{cosec}^2\left(\frac{k\pi}{2n+1}\right)$,

$\frac{2n(2n-1)}{6}<\left(\frac{2n+1}{\pi}\right)^2+\left(\frac{2n+1}{2\pi}\right)^2+\ldots+\left(\frac{2n+1}{n\pi}\right)^2<\frac{2n(2n+2)}{6}$.

$\frac{2n(2n-1)}{6}\frac{\pi^2}{(2n+1)^2}<\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}<\frac{2n(2n+2)}{6}\frac{\pi^2}{(2n+1)^2}$.

$\therefore \frac{\pi^2}{6}\left(\frac{2n}{2n+1}\right)\left(\frac{2n-1}{2n+1}\right)<\frac{1}{1^2}+\frac{1}{2^2}+\ \ldots+\frac{1}{n^2}<\frac{\pi^2}{6}\left(\frac{2n}{2n+1}\right)\left(\frac{2n+2}{2n+1}\right)$.

(vi) As $n\to\infty, \frac{2n}{2n+1}\to 1, \frac{2n-1}{2n+1}\to 1, \frac{2n+2}{2n+1}\to 1$, $\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}$ is sandwiched between two values which are the same, $\frac{\pi^2}{6}$, $\therefore \lim_{n\to\infty}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}\right)=\frac{\pi^2}{6}$.