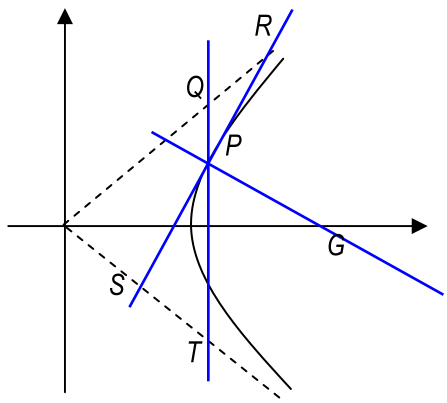
Practice Questions

Conical Sections



(a) (i) Let P be $(a\sec\theta, b\tan\theta)$

$$m=\frac{dy/d\theta}{dx/d\theta}=\frac{b\sec^2\theta}{a\sec\theta\tan\theta}=\frac{b\sec\theta}{a\tan\theta}$$

$$y-b\tan\theta=\frac{b\sec\theta}{a\tan\theta}(x-a\sec\theta)$$

$$a\tan\theta y-ab\tan^2\theta=b\sec\theta x-ab\sec^2\theta$$

$$b\sec\theta x-a\tan\theta y=ab\left(\sec^2\theta-\tan^2\theta\right)$$

$$b\sec\theta x-a\tan\theta y=ab$$

(ii) From the equation of the normal at P, $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$, let $y=0$,

$$\therefore x=\frac{\left(a^2+b^2\right)\sec\theta}{a}=\frac{a^2e^2\sec\theta}{a}=ae^2\sec\theta. \therefore G\left(ae^2\sec\theta,0\right)$$

Put $x=a\sec\theta$ into $y=\frac{bx}{a}$ gives $y=b\sec\theta. \therefore Q\left(a\sec\theta,b\sec\theta\right)$

$$m_{GQ}=\frac{b\sec\theta-0}{a\sec\theta-ae^2\sec\theta}=\frac{b}{a-ae^2}=\frac{b}{a(1-e^2)} \text{ and } m_{PR}=\frac{b}{a}.$$

$$\therefore m_{GQ}.m_{PR}=\frac{b^2}{a^2(1-e^2)}=\frac{b^2}{-b^2}=-1, \therefore \angle GQR=90°$$

(iii) $\angle GQR=\angle GPR=90°$, $\therefore GPQR$ is cyclic (equal angles subtending the same arc)

(iv) From the equation of the tangent, $b\sec\theta x-a\tan\theta y=ab$, let $y=\frac{bx}{a}$ gives

$$x=\frac{ab}{b\sec\theta-b\tan\theta}=\frac{a}{\sec\theta-\tan\theta}=\frac{a(\sec\theta+\tan\theta)}{\sec^2\theta-\tan^2\theta}=a(\sec\theta+\tan\theta).$$

$$y=\frac{bx}{a}=b(\sec\theta+\tan\theta). \therefore R\left(a(\sec\theta+\tan\theta),b(\sec\theta+\tan\theta)\right)$$

Also, from the equation of the tangent, $b\sec\theta x - a\tan\theta y = ab$, let $y = -\frac{bx}{a}$ gives

$$x = \frac{ab}{b\sec\theta + b\tan\theta} = \frac{a}{\sec\theta + \tan\theta} = \frac{a(\sec\theta - \tan\theta)}{\sec^2\theta - \tan^2\theta} = a(\sec\theta - \tan\theta).$$

$$y = -\frac{bx}{a} = b(\tan\theta - \sec\theta). \therefore S\left(a(\sec\theta - \tan\theta), b(\tan\theta - \sec\theta)\right)$$

Midpoint of RS has coordinates

$$\begin{cases} x = \dfrac{a(\sec\theta + \tan\theta) + a(\sec\theta - \tan\theta)}{2} = a\sec\theta \\ y = \dfrac{b(\sec\theta + \tan\theta) + b(\tan\theta - \sec\theta)}{2} = b\tan\theta \end{cases}$$

which are the coordinates of P, $\therefore PR = PS$.
(v) Q and T are symmetrical about the x-axis, so $\angle GTO = \angle GQO = 90°$, $\therefore \angle GTO + \angle GPS = 90° + 90° = 180°$, $\therefore GPST$ is cyclic (opposite angles are supplementary).
Further, GS is the diameter of this circle.
Similarly, GR is the diameter of the circle $GPQR$.
But $GS^2 = GP^2 + PS^2 = GP^2 + PR^2 = GR^2$, $\therefore GS = GR$. $\therefore$ The two circles have equal diameter, $\therefore$ they have the equal area.

(b) The equation of the chord of contact PQ is $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1, \therefore b^2x_0x + a^2y_0y - a^2b^2 = 0$.

This chord always touches the circle $x^2 + y^2 = b^2$, so the perpendicular distance from the centre of the circle (0,0) to the chord is equal to b.

$$\frac{a^2b^2}{\sqrt{b^4{x_0}^2 + a^4{y_0}^2}} = b$$

$$\therefore a^4b^2 = b^4{x_0}^2 + a^4{y_0}^2.$$

$\frac{{x_0}^2}{a^4} + \frac{{y_0}^2}{b^4} = \frac{1}{b^2}$, on dividing both sides by a^4b^4.

$\therefore$ The locus of T is the ellipse $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{b^2}$.

(c) The distance $PQ = 2OP = 2\sqrt{c^2p^2 + \frac{c^2}{p^2}} = \frac{2c}{p}\sqrt{p^4 + 1}$.

$\therefore$ The equation of the circle of centre P, radius PQ is

$$(x - cp)^2 + \left(y - \frac{c}{p}\right)^2 = \frac{4c^2}{p^2}\left(p^4 + 1\right)$$

Put $y = \frac{c^2}{x}$ gives $(x - cp)^2 + \left(\frac{c^2}{x} - \frac{c}{p}\right)^2 = \frac{4c^2}{p^2}\left(p^4 + 1\right)$

$$x^2 - 2cpx + c^2p^2 + \frac{c^4}{x^2} - \frac{2c^3}{px} + \frac{c^2}{p^2} = \frac{4c^2}{p^2}\left(p^4 + 1\right)$$

$$x^2 - 2cpx - 3c^2p^2 + \frac{c^4}{x^2} - \frac{2c^3}{px} - \frac{3c^2}{p^2} = 0$$

$$p^2x^4 - 2cp^3x^3 - 3c^2p^4x^2 - 3c^2x^2 - 2c^3px + c^4p^2 = 0 \qquad (1)$$

$$\therefore \sum\alpha = x_Q + x_D + x_E + x_F = \frac{2cp^3}{p^2} = 2cp,$$

$$\therefore x_D + x_E + x_F = 2cp - (-cp) = 3cp = 3x_P, \therefore x_P = \frac{x_D + x_E + x_F}{3}.$$

Replacing x in (1) by $\frac{c^2}{y}$ gives

$$\frac{p^2c^8}{y^4} - \frac{2p^3c^7}{y^3} - \frac{3p^4c^6}{y^2} - \frac{3c^6}{y^2} - \frac{2c^5p}{y} + c^4p^2 = 0$$

$$p^2c^4 - 2p^3c^3y - 3p^4c^2y^2 - 3c^2y^2 - 2cpy^3 + p^2y^4 = 0$$

$$\therefore \sum\alpha = y_Q + y_D + y_E + y_F = \frac{2cp}{p^2} = \frac{2c}{p}.$$

$$\therefore y_D + y_E + y_F = \frac{2c}{p} - y_Q = \frac{2c}{p} - \left(-\frac{c}{p}\right) = \frac{3c}{p} = 3y_P, \therefore y_P = \frac{y_D + y_E + y_F}{3}.$$