

**Multiple Choice**

- 1) (D)  $(2 \times 1 + 1) + (2 \times 2 + 1) + \dots + (2 \times 20 + 1)$   
 2) (B)  $P(2) = 2 \times 8 - 10 \times 4 + 6 \times 2 + 2 = -10$   
 3) (A)  $\tan x = \tan(2x - x)$   
 4) (C)  $360^\circ - (180^\circ + 40^\circ) = 140^\circ, \therefore$  the reflex angle  
 $= 220^\circ$   
 5) (A)  $= \int \frac{1 - \cos 4x}{2} dx$   
 6) (D)  $(2 \sin x - 1)(\sin x - 3) = 0, \sin x = \frac{1}{2},$   
 $\therefore x = n\pi + (-1)^n \frac{\pi}{6}$   
 7) (C)  $\dot{x} = 20 \cos 4t - 48 \sin 4t.$   
 Amplitude  $= \sqrt{20^2 + 48^2} = 52$   
 8) (B), At least  $1 = {}^{18}C_{11} - {}^{15}C_{11} = 30459$   
 9) (A)  $f(1) < 1, f'(x) > 1$  and  $f''(1) < 0$  (concave down)  
 10) (A)  $p'(x) = 3ax^2 + 2bx + c, p''(x) = 6ax + 2b.$   
 POI occurs at  $x = -\frac{b}{3a} < 0$ , since  $a > 0, b > 0$

**Question 11**

- (a)  $f: y = x^3 - 2$   
 $f^{-1}: x = y^3 - 2, \therefore y = \sqrt[3]{x+2}$   
 (b)  $u = x - 4, du = dx$   

$$\int (u+4)\sqrt{u} du = \int \left(u^{\frac{3}{2}} + 4u^{\frac{1}{2}}\right) du = \frac{2u^{\frac{5}{2}}}{5} + \frac{8u^{\frac{3}{2}}}{3} + C$$

$$= \frac{2\sqrt{(x-4)^5}}{5} + \frac{8\sqrt{(x-4)^3}}{3} + C$$
  
 (c)  $\frac{6}{1+4x^2}$   
 (d)  $\lim_{x \rightarrow 0} \frac{2 \sin x \cos x}{3x} = \lim_{x \rightarrow 0} \frac{\sin 2x}{3x} = \frac{2}{3} \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} = \frac{2}{3}$   
 (e)  $\frac{3}{2x+5} > x$   
 $3(2x+5) > x(2x+5)^2, x \neq -\frac{5}{2}$   
 $(2x+5)(2x^2+5x-3) < 0$   
 $(2x+5)(2x-1)(x+3) < 0$   
 $\therefore x < -3, -\frac{5}{2} < x < \frac{1}{2}$   
 (f) (i)  $P(\text{exactly once in 3 throws}) = {}^3C_1 \left(\frac{3}{5}\right) \left(\frac{2}{5}\right)^2 = \frac{36}{125}$   
 (ii)  $P(\text{at least 2 in 6 throws}) = 1 - P(x=0 \text{ or } 1)$   
 $= 1 - \left(\frac{2}{5}\right)^6 - {}^6C_1 \left(\frac{3}{5}\right) \left(\frac{2}{5}\right)^5 = \frac{2997}{3125}$

## Question 12

(a) (i) By similar triangles,  $\frac{r}{5} = \frac{h}{20}, \therefore r = \frac{h}{4}$

$$(ii) v = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h \frac{h^2}{16} = \frac{\pi}{48} h^3$$

$$\frac{dv}{dh} = \frac{\pi}{48} \times 3h^2 = \frac{\pi}{16} h^2$$

$$(iii) \text{ Given } \frac{dA}{dt} = -0.04 \text{ cm}^2 \text{ s}^{-1}, A = \pi r^2 = \frac{\pi}{16} h^2$$

$$\therefore \frac{dA}{dh} = \frac{\pi h}{8}$$

$$\frac{dh}{dt} = \frac{dh}{dA} \frac{dA}{dt} = \frac{8}{\pi h} \times -0.04 = \frac{-0.32}{\pi h}$$

$$(iv) \frac{dv}{dt} = \frac{dv}{dh} \frac{dh}{dt} = \frac{\pi}{16} h^2 \times \frac{-0.32}{\pi h} = -0.02h$$

$$\text{When } h = 10, \frac{dv}{dt} = -0.2 \text{ cm}^3 \text{ s}^{-1}$$

$$(b) (i) \frac{dx}{dt} = ky = k(500 - x)$$

$$\text{When } x = 0, \frac{dx}{dt} = 2 = 500k, \therefore k = \frac{2}{500} = 0.004$$

$$\therefore \frac{dx}{dt} = 0.004(500 - x)$$

$$(ii) x = 500 - Ae^{-0.004t}, \text{ i.e. } Ae^{-0.004t} = 500 - x$$

$$\frac{dx}{dt} = 0.004Ae^{-0.004t} = 0.004(500 - x)$$

$$\therefore x = 500 - Ae^{-0.004t} \text{ satisfies the equation.}$$

$$\text{When } t = 0, x = 0, \therefore A = 500.$$

(c) (i) For  $y = \tan x, y' = \sec^2 x$ . At  $x = \alpha, m_1 = \sec^2 \alpha$

For  $y = \cos x, y' = -\sin x$ . At  $x = \alpha, m_2 = -\sin \alpha$

When they meet at  $x = \alpha, \tan \alpha = \cos \alpha,$

$$\frac{\sin \alpha}{\cos \alpha} = \cos \alpha$$

$$\cos^2 \alpha = \sin \alpha.$$

$$m_1 m_2 = -\sec^2 \alpha \sin \alpha = -\frac{\sin \alpha}{\cos^2 \alpha} = -1, \therefore \text{the tangents}$$

are perpendicular.

$$(ii) f(x) = \tan x - \cos x,$$

$$f'(x) = \sec^2 x + \sin x$$

$$x_2 = 1 - \frac{f(1)}{f'(1)} = 0.76$$

## Question 13

(a) (i) High tide = 9, low tide = 1,  $\therefore$  amplitude = 4.

$$\text{Period} = \frac{25}{2} \text{ hours}, \therefore w = \frac{2\pi}{\text{Period}} = \frac{4\pi}{25}.$$

By choosing the origin when the tide is 5 m, and when  $t = 0$ , it's the high tide,  $x = 5 + 4 \cos\left(\frac{4\pi}{25}t\right)$ .

$$(ii) \frac{dx}{dt} = -\frac{16\pi}{25} \sin\left(\frac{4\pi}{25}t\right)$$

$\frac{dx}{dt}$  is maximum and positive when  $\sin\left(\frac{4\pi}{25}t\right) = -1$ .

$$\frac{4\pi}{25}t = \frac{3\pi}{2}, \therefore t = \frac{75}{8}, \text{ taking the earliest time}$$

$\therefore$  It's  $\frac{75}{8}$  hours after 2 am, i.e. 11:22:30 am.

(b) (i)  $\dot{y} = u \sin \theta - 10t$

$$\dot{y} = 0 \text{ when } t = \frac{u \sin \theta}{10}$$

$$\therefore y = \frac{u^2 \sin^2 \theta}{10} - 5 \frac{u^2 \sin^2 \theta}{100} = \frac{u^2 \sin^2 \theta}{20}.$$

$\therefore$  The greatest height is  $\frac{u^2 \sin^2 \theta}{20}$ .

(ii)  $u = 30, \theta = 30^\circ, \therefore$  The greatest height above the

$$\text{projection is } \frac{30^2 \sin^2 30^\circ}{20} = \frac{45}{4}.$$

$\therefore$  The height above the ground is  $\frac{45}{4} + 20 = \frac{125}{4}$  m

(iii) For the ball to rebound, it follows the reversed

path. Taking  $y = -5t^2 = \frac{-125}{4}$ , since  $\dot{y} = 0$

$$t^2 = \frac{125}{20} = \frac{25}{4}, \therefore t = \frac{5}{2} \text{ s.}$$

$$(iv) x = ut = 10 \times \frac{5}{2} = 25 \text{ m.}$$

(c) (i)  $\angle ACB = \angle ADB = 90^\circ$  (semi-circle angles)

$\therefore CMDE$  is a cyclic quad because opposite angles are supplementary.

(ii) In  $\triangle MDE$  and  $\triangle AEF$ ,

$\angle MED = \angle AEF$  (vertically opposite angles)

$\angle DME = \angle EAF$  (both =  $90^\circ - \angle DBA$ , angle sum in a  $\triangle$ )

$\therefore \triangle MDE \parallel \triangle AEF$  (AA)

$\therefore MF \perp AB$  (corresponding angles in similar  $\triangle$ s)

**Question 14**

(a) (i)  $\text{RHS} = 4n^3 + 14n^2 + 9n + 4n^2 + 14n + 9$   
 $= 4n^3 + 18n^2 + 23n + 9$

(ii) Let  $n = 1$ ,  $\text{LHS} = 1 \times 3 = 3$ ,  $\text{RHS} = \frac{1}{3} \times 1 \times 9 = 3$

The statement is true for  $n = 1$

Assume  $1 \times 3 + 3 \times 5 + \dots + (2n-1)(2n+1) =$

$$\frac{1}{3}n(4n^2 + 6n - 1), \text{ for some value of } n.$$

Required to prove

$$1 \times 3 + 3 \times 5 + \dots + (2n-1)(2n+1) + (2n+1)(2n+3)$$

$$= \frac{1}{3}(n+1)(4(n+1)^2 + 6(n+1) - 1)$$

$$\text{RHS} = \frac{1}{3}(n+1)(4n^2 + 8n + 4 + 6n + 6 - 1)$$

$$= \frac{1}{3}(n+1)(4n^2 + 14n + 9)$$

$$= \frac{1}{3}(4n^3 + 18n^2 + 23n + 9).$$

$$\text{LHS} = \frac{1}{3}n(4n^2 + 6n - 1) + (2n+1)(2n+3)$$

$$= \frac{1}{3}(4n^3 + 6n^2 - n) + (4n^2 + 8n + 3)$$

$$= \frac{1}{3}(4n^3 + 6n^2 - n + 12n^2 + 24n + 9)$$

$$= \frac{1}{3}(4n^3 + 18n^2 + 23n + 9) = \text{RHS}.$$

By the principle of Induction, it's true for all  $n \geq 1$ .

(b) (i)  $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$  (1)

Let  $x = 1$ ,  $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$ .

(ii) Differentiate both sides of (1) wrt  $x$ ,

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \dots + n\binom{n}{n}x^{n-1}$$

Let  $x = 1$ ,  $n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + n\binom{n}{n}$ .

(iii) Multiplying the result of (i) by  $n$  gives

$$n2^n = n \left( \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} \right) = \sum_{r=0}^n \binom{n}{r} n$$

$$\therefore n2^n - n = \sum_{r=1}^n \binom{n}{r} n \quad (2)$$

Multiplying the result of (ii) by 2 gives

$$2n2^{n-1} = 2 \left( \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + n\binom{n}{n} \right)$$

$$= \sum_{r=1}^n \binom{n}{r} 2r \quad (3)$$

(3) - (2) gives

$$\sum_{r=1}^n \binom{n}{r} (2r - n) = 2n2^{n-1} - (n2^n - n)$$

$$= n2^n - (n2^n - n) = n$$

(c) (i) The equation of the tangent is  $y = tx - at^2$

When  $y = -a$ ,  $tx = at^2 - a$ ,  $\therefore x = at - \frac{a}{t}$ .

$$\therefore D \left( at - \frac{a}{t}, -a \right)$$

(ii) The equation of the normal

$$y - at^2 = -\frac{1}{t}(x - 2at)$$

Sub  $x = at - \frac{a}{t}$  gives  $y - at^2 = \frac{1}{t} \left( at + \frac{a}{t} \right)$

$$y = a + \frac{a}{t^2} + at^2.$$

$$R \begin{cases} x = a \left( t - \frac{1}{t} \right) \\ y - a = a \left( t^2 + \frac{1}{t^2} \right) \end{cases}$$

Use  $\left( t - \frac{1}{t} \right)^2 = t^2 + \frac{1}{t^2} - 2$  gives

$$\frac{x^2}{a^2} = \frac{y-a}{a} - 2$$

$$\frac{x^2}{a^2} = \frac{y}{a} - 3$$

$\therefore$  The locus is the parabola  $x^2 = ay - 3a^2$

(iii) The focal length is  $\frac{a}{4}$

$$(iv) \text{ For } P_2, y = \frac{x^2 + 3a^2}{a}$$

$$\therefore y' = \frac{2x}{a}$$

$\therefore$  The gradient of the normal to  $P_2$  at  $R$  where

$$x = a\left(t - \frac{1}{t}\right) \text{ is } m = -\frac{a}{2x} = -\frac{1}{2\left(t - \frac{1}{t}\right)} = -\frac{t}{2(t^2 - 1)}$$

The 2 normals are identical when  $-\frac{t}{2(t^2 - 1)} = -\frac{1}{t}$

$$t^2 = 2(t^2 - 1)$$

$$t^2 = 2$$

$$t = \pm\sqrt{2}$$