Multiple choice questions

Q1 (C), $P(2)=8-16-12+k=0, \therefore k=20$

Q2 (D)

Q3 (C), $=\pi-\frac{1}{2}\times\frac{3\pi}{5}=\frac{7\pi}{10}$

Q4 (D)

Q5 (A)

Q6 (B)

Q7 (A)

Q8 (C), $\sin 2\theta=2\sin\theta\cos\theta=2\times\frac{5}{13}\times\frac{12}{13}=\frac{120}{169}$

Q9 (B)

Q10 (C), The solution is $-2\le x\le 3$

Question 11

(a) $\alpha\beta\gamma=-\frac{7}{2}$

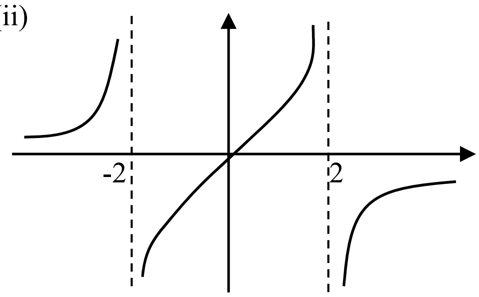
(b) $\int\frac{1}{\sqrt{49-4x^2}}dx=\frac{1}{2}\sin^{-1}\frac{2x}{7}+C$

(c) ${}^{10}C_7\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^3$

(d) (i) $f'(x)=\frac{(4-x^2)+2x^2}{(4-x^2)^2}=\frac{4+x^2}{(4-x^2)^2}>0$ since

$4+x^2>0$

(ii)



(e) $\lim\limits_{x\to 0}\frac{\sin\frac{x}{2}}{3x}=\lim\limits_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\times\frac{1}{6}=\frac{1}{6}$

(f) $\int_0^{\frac{1}{3}}\frac{e^{3x}}{e^{6x}+1}dx=\frac{1}{3}\left[\tan^{-1}e^{3x}\right]_0^{\frac{1}{3}}$

$=\frac{1}{3}\left(\tan^{-1}1-\tan^{-1}0\right)=\frac{1}{3}\frac{\pi}{4}=\frac{\pi}{12}$

(g) $\frac{d}{dx}\left(x^2\sin^{-1}5x\right)=2x\sin^{-1}5x+\frac{5x^2}{\sqrt{1-25x^2}}$

Question 12

(a) (i) $\sqrt{3}\cos x-\sin x=2\cos\left(x+\frac{\pi}{6}\right)$

(ii) $\sqrt{3}\cos x-\sin x=1$

$2\cos\left(x+\frac{\pi}{6}\right)=1$

$\cos\left(x+\frac{\pi}{6}\right)=\frac{1}{2}$

$x+\frac{\pi}{6}=\pm\frac{\pi}{3}+k2\pi, k\in Z$

$x=\frac{\pi}{6}+k2\pi, -\frac{\pi}{2}+k2\pi$

$\therefore$ For $0\le x\le 2\pi, x=\frac{\pi}{6},\frac{3\pi}{2}$

(b) $V=\pi\int_0^{\frac{3\pi}{2}}9\sin^2\frac{x}{2}dx=\frac{9\pi}{2}\int_0^{\frac{3\pi}{2}}(1-\cos x)dx$

$=\frac{9\pi}{2}\left[x-\sin x\right]_0^{\frac{3\pi}{2}}=\frac{9\pi}{2}\left(\frac{3\pi}{2}+1\right)$ u^3

(c) $T=A+Be^{-kt}$ where $A=22°$

When $t=0, T=80°, \therefore 80°=22°+B, \therefore B=58°$

When $t=10, T=60°, \therefore 60°=22°+58°e^{-10k}$

$\therefore e^{-10k}=\frac{38}{58}=\frac{19}{29}$

$\therefore k=-\frac{1}{10}\ln\frac{19}{29}$

$\therefore T=22+58e^{\left(\frac{1}{10}\ln\frac{19}{29}\right)t}$

When $T=40, 40=22+58e^{\left(\frac{1}{10}\ln\frac{19}{29}\right)t}$

$e^{\left(\frac{1}{10}\ln\frac{19}{29}\right)t}=\frac{18}{58}=\frac{9}{29}$

$\left(\frac{1}{10}\ln\frac{19}{29}\right)t=\ln\frac{9}{29}$

$\therefore t=27.67\simeq 28$ minutes.

(d) (i) The equation of (ℓ) is $2x-y-1=0$

$D(t)=\frac{\left|2t-(t^2+3)-1\right|}{\sqrt{2^2+1^2}}=\frac{\left|-t^2+2t-4\right|}{\sqrt{5}}=\frac{t^2-2t+4}{\sqrt{5}}$

since $t^2-2t+4=(t-1)^2+3>0$.

(ii) $\frac{dD}{dt}=\frac{1}{\sqrt{5}}(2t-2)$

$\frac{dD}{dt}=0$ when $t=1$

$\frac{d^2D}{dt^2}=\frac{2}{\sqrt{5}}>0, \therefore$ minimum when $t=1$.

(iii) The gradient of the tangent at P is $m=\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$=\frac{2t}{1}$. When $t=1$, $m=2$, which is the same gradient of (ℓ). $\therefore$ The tangent is parallel to (ℓ).

(e) $v^2=k-9x^2$

$\ddot{x}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)=\frac{1}{2}\frac{d}{dx}\left(k-9x^2\right)=-9x.$

$\therefore$ SHM, with $n=3$

Period $=\frac{2\pi}{n}=\frac{2\pi}{3}$.

Question 13

(a) (i) $V=\frac{4}{3}\pi r^3, \therefore \frac{dV}{dr}=4\pi r^2=A$

$\frac{dr}{dt}=\frac{dr}{dV}\frac{dV}{dt}=\frac{1}{A}\times -10^{-4}A=-10^{-4}$

(ii) $r=-10^{-4}t+C$

When $t=0, V=10^{-6}, r=\sqrt[3]{\frac{3\times 10^{-6}}{4\pi}}, \therefore C=\sqrt[3]{\frac{3\times 10^{-6}}{4\pi}}$

$\therefore r=-10^{-4}t+\sqrt[3]{\frac{3\times 10^{-6}}{4\pi}}$

When $r=0, t=\frac{\sqrt[3]{\frac{3\times 10^{-6}}{4\pi}}}{10^{-4}}=62$ s.

(b) (i) $x=\frac{2\times ap-0}{2-1}=2ap,$

$y=\frac{2\times 0-(2a+ap^2)}{2-1}=-2a-ap^2$

(ii) $x=2ap, \therefore p=\frac{x}{2a}$.

$y=-2a-a\frac{x^2}{4a^2}=-2a-\frac{x^2}{4a}$

$-4ay-8a^2=x^2$

$-4a(y+2a)=x^2$

Focal length $=a$, Vertex $y=-2a$, Directrix $y=-a$

$\therefore$ The same directrix and focal length as $4ay=x^2$.

(c) (i) The particle fired from A has equations

$x=ut\cos\alpha, y=ut\sin\alpha-\frac{gt^2}{2}$, if the origin is chosen at A.

$\therefore \dot{y}=u\sin\alpha-gt$

Maximum height is reached when $\dot{y}=0$,

$\therefore$ when $t=\frac{u\sin\alpha}{g}$.

(ii) The particle fired from B has equations

$x=wt\cos\beta, y=wt\sin\beta-\frac{gt^2}{2}$, if the origin is chosen at B.

$\therefore$ It reaches maximum height when $t=\frac{w\sin\beta}{g}$.

$\therefore$ When they both reach maximum height,

$\frac{u\sin\alpha}{g}=\frac{w\sin\beta}{g}$

$\therefore u\sin\alpha=w\sin\beta$

(iii) d is the distance travelled by both particles,

$d=ut\cos\alpha+wt\cos\beta$

$=\frac{u^2\sin\alpha\cos\alpha}{g}+\frac{w^2\sin\beta\cos\beta}{g}$

$=\frac{u\cos\alpha}{g}w\sin\beta+\frac{w\cos\beta}{g}u\sin\alpha$

$=\frac{uw}{g}\left(\cos\alpha\sin\beta+\sin\alpha\cos\beta\right)$

$=\frac{uw}{g}\sin(\alpha+\beta)$

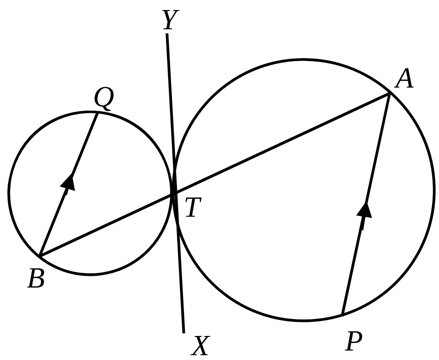
(d) Draw the common tangent XY to the circles at T

$\angle TAP=\angle XTP$ and $\angle QBT=\angle YTQ$ (angle between a chord and a tangent is equal to the angle subtending the alternate segment)

But $\angle TAP=\angle QBT$ (alternate angles on parallel lines)

$\therefore \angle XTP=\angle YTQ$

$\therefore Q, T, P$ are collinear (vertically opposite angles)



Question 14

(a) (i) For $k>0, (k+1)^2 > (k(k+1)$

$$\frac{1}{(k+1)^2} < \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$\therefore \frac{1}{(k+1)^2} - \frac{1}{k} + \frac{1}{k+1} < 0$$

(ii) Let $n=2, \text{LHS} = \frac{1}{1^2} + \frac{1}{2^2} = \frac{5}{4}, \text{RHS} = 2 - \frac{1}{2} = \frac{3}{2}$

$\therefore$ true, since $\frac{5}{4} < \frac{3}{2}$.

Assume $\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for some integer n

RTP $\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2} + \frac{1}{(n+1)^2} < 2 - \frac{1}{n+1}$

$$\text{LHS} < 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$$

$$< 2 - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1}, \text{ from part (i)}$$

$$= 2 - \frac{1}{n+1} = \text{RHS}, \therefore \text{true.}$$

$\therefore$ By the principle of Induction, it's true for all $n \geq 2$

(b) (i) $\binom{4n}{2n}$

(ii) $\left(1+(x^2+2x)\right)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} 1^k (x^2+2x)^{2n-k}$

$= \sum_{k=0}^{2n} \binom{2n}{k} x^{2n-k}(x+2)^{2n-k}$, as $x^2+2x = x(x+2)$.

(iii) $1+x^2+2x = (1+x)^2, \therefore \left(1+x^2+2x\right)^{2n} = (1+x)^{4n}$

$\therefore \binom{4n}{2n}$ is the coefficient of x^{2n} in $(1+x)^{4n}$.

Given $x^{2n-k}(x+2)^{2n-k} = \binom{2n-k}{0} 2^{2n-k} x^{2n-k} +$

$\binom{2n-k}{1} 2^{2n-k-1} x^{2n-k+1} + \ldots + \binom{2n-k}{2n-k} 2^0 x^{4n-2k}$,

$\sum_{k=0}^{2n} \binom{2n}{k} x^{2n-k}(x+2)^{2n-k} = \sum_{k=0}^{2n} \binom{2n}{k} \times$

$\left\{ \binom{2n-k}{0} 2^{2n-k} x^{2n-k} + \binom{2n-k}{1} 2^{2n-k-1} x^{2n-k+1} + \ldots \right.$

$\left. + \binom{2n-k}{2n-k} 2^0 x^{4n-2k} \right\}$

In $\binom{2n-k}{0} 2^{2n-k} x^{2n-k} + \binom{2n-k}{1} 2^{2n-k-1} x^{2n-k+1} +$

$+ \ldots + \binom{2n-k}{2n-k} 2^0 x^{4n-2k}$, the coefficient of x^{2n} is

$\binom{2n}{0} 2^{2n}$, when $k=0$,

$+ \binom{2n-1}{1} 2^{2n-2}$, when $k=1$,

$+ \ldots + \binom{2n-n}{n} 2^0$, when $k=n$.

$\therefore$ The coefficient of x^{2n} is

$\binom{2n}{0}\binom{2n}{0} 2^{2n} + \binom{2n}{1}\binom{2n-1}{1} 2^{2n-2} + \ldots$

$+ \binom{2n}{n}\binom{2n-n}{n} 2^0$.

$$\therefore \binom{4n}{2n} = \sum_{k=0}^{n} \binom{2n}{k}\binom{2n-k}{k} 2^{2n-2k}.$$

(c) (i) Let $f(t) = e^t - \frac{1}{t}$

$$f'(t) = e^t + \frac{1}{t^2}$$

$$t_1 = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.5 - \frac{e^{0.5}-2}{e^{0.5}+4} = 0.562 \approx 0.56$$

(ii) When two curves $g(x)$ and $h(x)$ have a common tangent at their point of intersection, $g(x) - h(x)$ yields a double root at the point of contact.

Let $f(x) = e^x - \ln(x) + \text{C}$

$$f'(x) = e^x - \frac{1}{x}.$$

From (i), when $x \approx 0.56$, $f'(x) \approx 0$

$\therefore$ We want $f(0.56) \approx 0$, too, $e^{0.56} - \ln 0.56 + C = 0$

$\therefore C = \ln 0.56 - e^{0.56} \approx -2.33$

$\therefore f(x) = e^x - \ln x - 2.33$

Let $2.33 = \ln C, \therefore C = e^{2.33} \approx 10.3$,

$\therefore f(x) = e^x - \ln(10.3x)$

$\therefore$ The curves e^x and $\ln(10.3x)$ have a common tangent at their point of intersection $x \approx 0.56$.

$\therefore$ The curves $e^{\frac{x}{10.3}}$ and $\ln x$ have a common tangent at their point of intersection $x \approx 5.77$, i.e. $r \approx \frac{1}{10.3}$

$= 0.097$