2010 Extension 2 Solution

Question 1

(a) $\int \frac{xdx}{\sqrt{1+3x^2}} = \frac{1}{3}\sqrt{1+3x^2} + C$

(b) $\int_0^{\frac{\pi}{4}} \tan x\, dx = \left[\ln(\sec x)\right]_0^{\frac{\pi}{4}} = \ln\sqrt{2}.$

(c) $\int \frac{1}{x(x^2+1)}dx = \int\left(\frac{1}{x} + \frac{-x}{x^2+1}\right)dx = \ln x - \frac{1}{2}\ln(x^2+1) + C$

(d) Let $t = \tan\frac{x}{2}, dt = \frac{1}{2}\sec^2\frac{x}{2}dx, \therefore dx = \frac{2dt}{\sec^2\frac{x}{2}} = \frac{2dt}{1+t^2}$

When $x = 0, t = 0$; when $x = \frac{\pi}{2}, t = 1$.

$I = \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}} \frac{2dt}{1+t^2} = \int_0^1 \frac{2dt}{(1+t)^2} = \left[\frac{-2}{1+t}\right]_0^1 = -1+2 = 1$

(e) Let $u^2 = x, 2udu = dx$

$I = \int \frac{2udu}{1+u} = \int \frac{2(u+1-1)}{1+u}du = 2\int\left(1 - \frac{1}{1+u}\right)du$

$= 2\left(u - \ln(1+u)\right) + C = 2\left(\sqrt{x} - \ln(1+\sqrt{x})\right) + C$

Question 2

(a) (i) $z^2 = (5-i)^2 = 25 - 1 - 10i = 24 - 10i$

(ii) $z + 2\overline{z} = 5 - i + 2(5+i) = 15 + i$

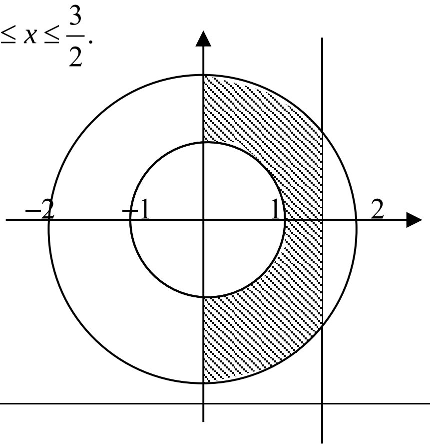
(iii) $\frac{i}{z} = \frac{i}{5-i} = \frac{i(5+i)}{26} = \frac{-1+5i}{26}$

(b) (i) $-\sqrt{3} - i = 2\left(\cos\frac{-5\pi}{6} + i\sin\frac{-5\pi}{6}\right)$

(ii) $\left(-\sqrt{3}-i\right)^6 = 2^6\left(\cos(-5\pi) + i\sin(-5\pi)\right) = -64$

(c) $z + \overline{z} = (x+iy) + (x-iy) = 2x, \therefore 0 \le z + \overline{z} \le 3 \Leftrightarrow$

$0 \le x \le \frac{3}{2}$.



(d) (i) $OA = OB = 1, \therefore$ A parallelogram with adjacent sides equal is a rhombus.

(ii) $\arg(z^2) = 2\arg z = 2\theta, \therefore \angle AOB = \theta$. In a rhombus the diagonals bisect, $\therefore \angle COA = \frac{\theta}{2}$

$\therefore \arg\left(z + z^2\right) = \theta + \frac{\theta}{2} = \frac{3\theta}{2}$.

(iii) Let M be the midpoint of $OC, \angle AMO = 90°$ (the diagonals are perpendicular)

$OC = 2OM = 2\cos\frac{\theta}{2}, \therefore \left|z + z^2\right| = 2\cos\frac{\theta}{2}$.

(iv) $z + z^2 = \cos\theta + i\sin\theta + \left(\cos\theta + i\sin\theta\right)^2 = \cos\theta + i\sin\theta + \cos 2\theta + i\sin 2\theta$ (de Moivre's theorem)

$\therefore \mathrm{Re}(z + z^2) = \cos\theta + \cos 2\theta$.

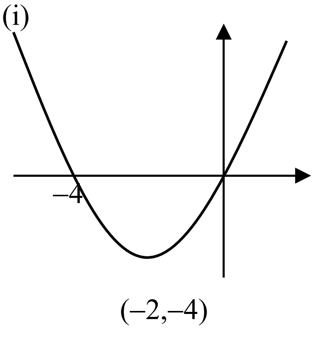
From (ii) and (iii), $z + z^2 = 2\cos\frac{\theta}{2}\left(\cos\frac{3\theta}{2} + i\sin\frac{3\theta}{2}\right)$

$\therefore \cos\theta + \cos 2\theta = 2\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$, by equating the real parts

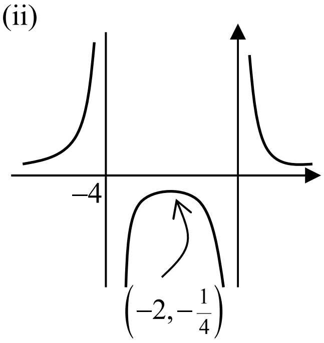
Question 3

(a)

(i)



(ii)



(b) Let's shift the curve to the left 1 unit, its new equation is $y = 1 - x^2$, and the axis of rotation is $x = 3$.

The ring at the height y has radii $3 - x$ and $3 + x$, so its area = $\pi\left(3+x\right)^2 - \pi\left(3-x\right)^2 = 12\pi x$.

$\partial V = 12\pi x \partial y = 12\pi\sqrt{1-y}\partial y.$

$$V = 12\pi\int_0^1 \sqrt{1-y}dy = -12\pi\left[\frac{2\sqrt{(1-y)^3}}{3}\right]_0^1 = 8\pi.$$

(c) Let $p = \Pr(H)$.
Given $\Pr(H)\Pr(T) + \Pr(T)\Pr(H) = 0.48$,

$2p(1-p) = 0.48$

$p^2 - p + 0.24 = 0$

$(p-0.6)(p-0.4) = 0.$

$\therefore p = 0.6$ or 0.4.

Since $p > 0.5$, $p = 0.6$.

$\therefore \Pr(H)\Pr(H) = 0.6^2 = 0.36.$

(d) (i) The gradient of $QA = \dfrac{c+\frac{c}{t}}{c+ct} = \dfrac{1}{t}.$

Eqn of ℓ_1 is $y - \dfrac{c}{t} = -t(x-ct)$

$y = \dfrac{c}{t} - tx + ct^2$

$= -tx + c\left(t^2 + \dfrac{1}{t}\right)$ (1)

(ii) $y = tx + c\left(t^2 - \dfrac{1}{t}\right).$

(iii) $-tx + c\left(t^2 + \dfrac{1}{t}\right) = tx + c\left(t^2 - \dfrac{1}{t}\right)$

$2tx = c\left(t^2 + \dfrac{1}{t} - t^2 + \dfrac{1}{t}\right) = \dfrac{2c}{t}$

$\therefore x = \dfrac{c}{t^2}.$

Sub. to (1), $y = \dfrac{c}{t} + c\left(t^2 - \dfrac{1}{t}\right) = ct^2.$

$\therefore P\left(\dfrac{c}{t^2}, ct^2\right).$

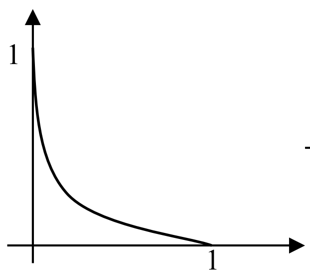
(iv) Since $\dfrac{c}{t^2}\times ct^2 = c^2$, the locus of P is also the hyperbola $xy = c^2$.

Question 4

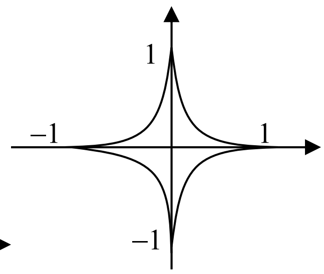
(a) (i) $\dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx} = 0$

$\therefore \dfrac{dy}{dx} = -\dfrac{\sqrt{y}}{\sqrt{x}}.$

(ii)



(iii)



(b) (i) Resolving the forces

vertically, $N\cos\alpha + F\sin\alpha = mg$ (1)

horizontally, $N\sin\alpha - F\cos\alpha = \dfrac{mv^2}{r}$ (2)

$(1)\times\sin\alpha \Rightarrow N\cos\alpha\sin\alpha + F\sin^2\alpha = mg\sin\alpha$ (3)

$(2)\times\cos\alpha \Rightarrow N\sin\alpha\cos\alpha - F\cos^2\alpha = \dfrac{mv^2}{r}\cos\alpha$ (4)

$(3)-(4)$ gives $F = mg\sin\alpha - \dfrac{mv^2}{r}\cos\alpha$

(ii) When $F = 0, mg\sin\alpha = \dfrac{mv^2}{r}\cos\alpha.$

$\therefore v^2 = gr\tan\alpha.$

(c) For all $a > 0, b > 0$,

$(a-b)^2 \geq 0, \therefore a^2 + b^2 \geq 2ab.$

$(a+b)^2 = a^2 + b^2 + 2ab \geq 2ab + 2ab = 4ab.$

$\dfrac{a+b}{ab} \geq \dfrac{4}{a+b}$

$\dfrac{1}{a} + \dfrac{1}{b} \geq \dfrac{4}{a+b}$

$\therefore \dfrac{1}{a} + \dfrac{1}{b} = \dfrac{k}{a+b}$, where $k \geq 4$.

(d) (i) ${}^{12}C_8$

(ii) $\dfrac{{}^{12}C_4 \times {}^{8}C_4}{3!}$, as there are 3! ways to arrange these 3 groups.

Question 5

(a) (i) $B(b\cos\theta, b\sin\theta)$

(ii) P is given by $(a\cos\theta, b\sin\theta)$

Sub. to $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ gives

$\text{LHS}=\cos^2\theta+\sin^2\theta=1=\text{RHS}.$

(iii) $m=\dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\dfrac{b\cos\theta}{-a\sin\theta}$

$y-b\sin\theta=\dfrac{b\cos\theta}{-a\sin\theta}(x-a\cos\theta)$

$-ay\sin\theta+ab\sin^2\theta=bx\cos\theta-ab\cos^2\theta.$

$bx\cos\theta+ay\sin\theta=ab(\cos^2\theta+\sin^2\theta).$

$bx\cos\theta+ay\sin\theta=ab.$ (1)

(iv) The equation of the tangent to the circle at A

is $ax\cos\theta+ay\sin\theta=a^2.$ (2)

$(1)\times a-(2)\times b$ gives $(a^2-ab)y\sin\theta=a^2b-a^2b.$

$\therefore y=0$

$\therefore$ The tangents meet on the x-axis.

(b) $\dfrac{1}{y(1-y)}=\dfrac{1}{y}+\dfrac{1}{1-y}$

$\therefore \displaystyle\int\frac{dy}{y(1-y)}=\int\left(\frac{1}{y}+\frac{1}{1-y}\right)dy=\ln y-\ln(1-y)+C$

$=\ln\dfrac{y}{1-y}+C.$

(c) (i) Let $\dfrac{dy}{dx}=f(y)=a(y-y^2).$

$y-y^2$ is an upside down parabola, its max occurs

when $f'(y)=0, \therefore 1-2y=0, \therefore\ y=\dfrac{1}{2}.$

(ii) $\dfrac{dy}{y(1-y)}=a\,dx$

$\ln\dfrac{y}{1-y}+\ln k=ax,$ using (b) and letting $C=\ln k$

$\ln\dfrac{ky}{1-y}=ax$

$\dfrac{ky}{1-y}=e^{ax}$

$ky+ye^{ax}=e^{ax}.$

$ke^{-ax}y+y=1.$

$y\left(ke^{-ax}+1\right)=1.$

$\therefore y=\dfrac{1}{ke^{-ax}+1}.$

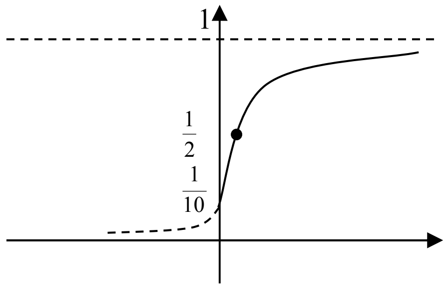
(iii) If $x\to\infty, e^{-ax}\to 0, \therefore y\to 1.$

If $x=0, \dfrac{1}{k+1}=\dfrac{1}{10}\times 1.$

$\therefore k=9.$

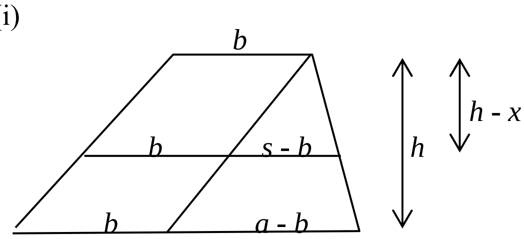
(iv) It's the point of inflexion at $y=\dfrac{1}{2}.$

(v)



Question 6

(a) (i)



Using similar triangles,

$\dfrac{h-x}{h}=\dfrac{s-b}{a-b}$

$\left(1-\dfrac{x}{h}\right)(a-b)=s-b$

$\therefore s=b+\left(1-\dfrac{x}{h}\right)(a-b)$

$=b+a-b-\dfrac{x}{h}(a-b)$

$=a-\dfrac{(a-b)x}{h}.$

(ii) $A=\left(a-\dfrac{(a-b)x}{h}\right)^2.$

$V=\displaystyle\int_0^h\left(a-\frac{(a-b)x}{h}\right)^2dx$

$=\left[\dfrac{-h}{3(a-b)}\left(a-\dfrac{(a-b)x}{h}\right)^3\right]_0^h$

$=\dfrac{-h}{3(a-b)}\left(b^3-a^3\right)$

$=\dfrac{h}{3}\left(a^2+ab+b^2\right).$

(b) When $n=0, a_0=\left(1+\sqrt{2}\right)^0+\left(1-\sqrt{2}\right)^0=2.$

When $n=1, a_1=\left(1+\sqrt{2}\right)+\left(1-\sqrt{2}\right)=2.$

$\therefore$ True for $n=0$ and 1.

Assume $a_{n-1}=\left(1+\sqrt{2}\right)^{n-1}+\left(1-\sqrt{2}\right)^{n-1}$ and $a_{n-2}=$ $\left(1+\sqrt{2}\right)^{n-2}+\left(1-\sqrt{2}\right)^{n-2}.$

RTP $a_n=\left(1+\sqrt{2}\right)^n+\left(1-\sqrt{2}\right)^n$

$a_n=2a_{n-1}+a_{n-2}$

$=2\left(1+\sqrt{2}\right)^{n-1}+2\left(1-\sqrt{2}\right)^{n-1}+\left(1+\sqrt{2}\right)^{n-2}+\left(1-\sqrt{2}\right)^{n-2}$

$=\left(1+\sqrt{2}\right)^{n-2}\left(2+2\sqrt{2}+1\right)+\left(1-\sqrt{2}\right)^{n-2}\left(2-2\sqrt{2}+1\right)$

$=\left(1+\sqrt{2}\right)^{n-2}\left(3+2\sqrt{2}\right)+\left(1-\sqrt{2}\right)^{n-2}\left(3-2\sqrt{2}\right)$

$=\left(1+\sqrt{2}\right)^{n-2}\left(1+\sqrt{2}\right)^2+\left(1-\sqrt{2}\right)^{n-2}\left(1-\sqrt{2}\right)^2$

$=\left(1+\sqrt{2}\right)^n+\left(1-\sqrt{2}\right)^n.$

$\therefore$ It's true for all $n\ge 0$.

(c) (i) $(\cos\theta+i\sin\theta)^5=c^5+5c^4(is)+10c^3(is)^2+$ $10c^2(is)^3+5c(is)^4+(is)^5$, where $c=\cos\theta, s=\sin\theta$,

$=\left(c^5-10c^3s^2+5cs^4\right)+i\left(5c^4s-10c^2s^3+s^5\right)$

(ii) By de Moivre's theorem,

$(\cos\theta+i\sin\theta)^5=\cos5\theta+i\sin5\theta.$

$\therefore \sin5\theta=5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$

$=5\left(1-\sin^2\theta\right)^2\sin\theta-10(1-\sin^2\theta)\sin^3\theta+\sin^5\theta$

$=5\left(1-2\sin^2\theta+\sin^4\theta\right)\sin\theta-10(1-\sin^2\theta)\sin^3\theta+\sin^5\theta$

$=5\sin\theta-10\sin^3\theta+5\sin^5\theta-10\sin^3\theta+10\sin^5\theta+\sin^5\theta$

$=16\sin^5\theta-20\sin^3\theta+5\sin\theta.$

(iii) Solving $\sin5\theta=1$ gives $5\theta=\frac{\pi}{2}+k\pi, \therefore \theta=\frac{\pi}{10}$ is one solution of $\sin5\theta=1, \therefore \sin\frac{\pi}{10}$ is one solution of $16x^5-20x^3+5x=1$, or $16x^5-20x^3+5x-1=0.$

(iv) $16x^5-20x^3+5x-1=(x-1)(16x^4+16x^3-4x^2-4x+1)$ by inspection method.

$\therefore\ p(x)=(16x^4+16x^3-4x^2-4x+1).$

(v) $a=2$, by equating the coefficients of $x^3, 8a=16.$

(vi) Solving $4x^2+2x-1=0$ gives $x=\frac{-2\pm\sqrt{20}}{8}=\frac{-1\pm\sqrt{5}}{4}$

$\therefore \sin\frac{\pi}{10}=\frac{-1+\sqrt{5}}{4}$, since $\frac{\pi}{10}$ lies in the 1st quadrant.

Question 7

(a) (i) $\angle ABD=\angle KCD$ (angles subtending same arc)

$\angle ADB=\angle KDC$ ($\angle ADB=\angle KDB+\angle ADK, \angle KDC=$ $\angle KDB+\angle BDC$ and $\angle ADK=\angle BDC$)

$\therefore \Delta ABD \parallel\!\mid \Delta KCD$ (equiangular)

(ii) $\frac{\Delta ABD}{\Delta KCD}\Rightarrow\frac{AB}{KC}=\frac{BD}{CD}$ (corresponding sides in similar triangles)

$\therefore AB\times CD=KC\times BD.$ (1)

Similarly, $\frac{\Delta AKD}{\Delta BCD}\Rightarrow\frac{AK}{BC}=\frac{AD}{BD},$

$\therefore BC\times AD=AK\times BD.$ (2)

$(1)+(2)$ gives

$AB\times CD+BC\times AD=(KC+AK)\times BD$

$=AC\times BD.$

(iii) Let $AC=BD=AB=x, AD=CD=BC=1,$

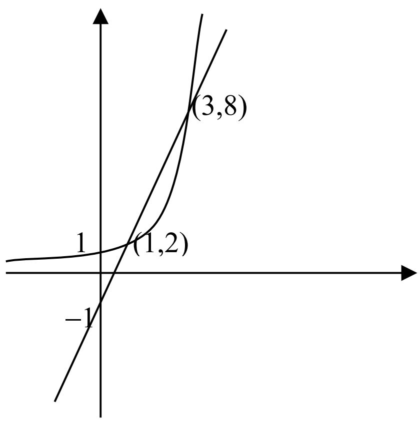
$x+1=x^2.$

$x^2-x-1=0.$

$x=\frac{1\pm\sqrt{5}}{2}.$

$\therefore x=\frac{1+\sqrt{5}}{2}$, since $x>0$.

(b)



(c) (i) $P'(x)=n(n-1)x^{n-1}-n(n-1)x^{n-2}$

$=n(n-1)x^{n-2}(x-1)$

$=0$ when $x=0$ or 1.

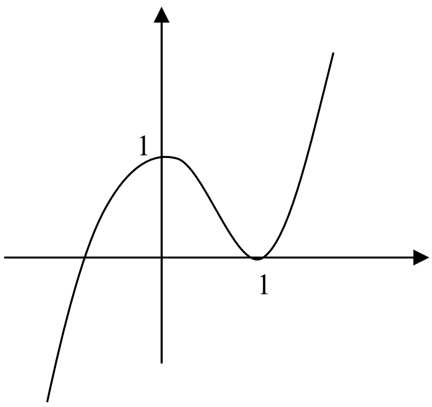
$\therefore$ It has two stationary points.

(ii) $P(1)=n-1-n+1=0$ and $P'(1)=0.$

$\therefore x=1$ is a double zero.

(iii) As $x\to+\infty, x^n\to+\infty$ but as $x\to-\infty, x^n\to-\infty$

$\therefore$ it has one real zero other than the double zero 1. Further, as it has only 2 turning points, $P(x)$ has exactly one real zero other than 1.



(iv) For $n \geq 3$, it can be proven that $P(x) = (n-1)x^n$

$-nx^{n-1} + 1$ has a zero between -1 and $-\frac{1}{2}$.

Substitute x by -1, noting that n is odd,

$P(-1) = (n-1)(-1) - n + 1 = -2n + 2 < 0$, since $n > 3$.

Substitute x by $-\frac{1}{2}$, noting that n is odd,

$P\left(-\frac{1}{2}\right) = (n-1)\frac{-1}{2^n} - n\frac{1}{2^{n-1}} + 1 = \frac{1}{2^n}\left(-n+1-2n\right) + 1$

$= \frac{-3n+1}{2^n} + 1.$ (1)

From (b), $2^n \geq 3n-1$ for $n \geq 3$,

$\therefore \frac{3n-1}{2^n} \leq 1$ for $n \geq 3, \therefore \frac{-3n+1}{2^n} \geq -1$

$\therefore$ (1) becomes > 0.

$P(x)$ is a continuous function, so when it changes signs, it must cut the x-axis.

$\therefore -1 < \alpha < -\frac{1}{2}$.

The equality occurs when $n = 3$, because from (1),

$P\left(-\frac{1}{2}\right) = \frac{-8}{8} + 1 = 0.$

$\therefore -1 < \alpha \leq -\frac{1}{2}$.

(v) Let the zeros of $4x^5 - 5x^4 + 1$ be $1, 1, \alpha, a \pm ib$.

Product of roots $= (a^2 + b^2)\alpha = -\frac{1}{4}$.

$\therefore a^2 + b^2 = -\frac{1}{4\alpha}$.

As $-1 < \alpha \leq -\frac{1}{2}, \frac{1}{4} < a^2 + b^2 \leq \frac{1}{2}, \therefore$ The modulus of all zeros is less than or equal to 1, noting that $|\alpha| < 1$, too.

Question 8

(a) Let $u = \cos^{2n-1} x, du = (2n-1)\cos^{2n-2} x(-\sin x)dx$

and $dv = \cos x\, dx, v = \sin x$.

$$A_n = \left[\cos^{2n-1} x \sin x\right]_0^{\frac{\pi}{2}} + (2n-1)\int_0^{\frac{\pi}{2}} \cos^{2n-2} x \sin^2 x dx$$

$$= (2n-1)\int_0^{\frac{\pi}{2}} \cos^{2n-2} x\left(1 - \cos^2 x\right)dx$$

$$= (2n-1)\int_0^{\frac{\pi}{2}} \cos^{2n-2} x dx - (2n-1)\int_0^{\frac{\pi}{2}} \cos^{2n} x dx$$

$$= (2n-1)A_{n-1} - (2n-1)A_n$$

$\therefore 2nA_n = (2n-1)A_{n-1}$.

$\therefore nA_n = \frac{2n-1}{2}A_{n-1}$.

(b) Let $u = \cos^{2n} x, du = 2n\cos^{2n-1} x(-\sin x)dx$

and $dv = dx, v = x$.

$$A_n = \left[x\cos^{2n} x\right]_0^{\frac{\pi}{2}} + 2n\int_0^{\frac{\pi}{2}} x\cos^{2n-1} x \sin x dx$$

$$= 2n\int_0^{\frac{\pi}{2}} x \sin x \cos^{2n-1} x dx.$$

(c) For $\int_0^{\frac{\pi}{2}} x\sin x\cos^{2n-1} x dx$, let $u = \sin x \cos^{2n-1} x$,

$du = \left(\cos^{2n} x - (2n-1)\cos^{2n-2} x \sin^2 x\right)dx$

$= \left(\cos^{2n} x - (2n-1)\cos^{2n-2} x(1-\cos^2 x)\right)dx$

$= \left(2n\cos^{2n} x - (2n-1)\cos^{2n-2} x\right)dx,$

and $dv = xdx, \therefore v = \frac{x^2}{2}$.

$$A_n = 2n\left[\frac{x^2}{2}\sin x \cos^{2n-1} x\right]_0^{\frac{\pi}{2}}$$

$$-2n\int_0^{\frac{\pi}{2}} \frac{x^2}{2}\left(2n\cos^{2n} x - (2n-1)\cos^{2n-2} x\right)dx$$

$$= -2n^2\int_0^{\frac{\pi}{2}} x^2\cos^{2n} x dx + (2n-1)n\int_0^{\frac{\pi}{2}} x^2 \cos^{2n-2} x dx$$

$$= -2n^2 B_n + (2n-1)nB_{n-1}.$$

$\therefore \frac{A_n}{n^2} = \frac{2n-1}{n}B_{n-1} - 2B_n.$ (1)

(d) From (a), $\frac{2A_n}{A_{n-1}} = \frac{2n-1}{n}$,

$(1) \Rightarrow \frac{A_n}{n^2} = \frac{2A_n}{A_{n-1}}B_{n-1} - 2B_n.$

$\therefore \frac{1}{n^2} = 2\left(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n}\right).$

(e) $\sum_{k=1}^{n}\frac{1}{k^2}=\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}$

$=2\left(\frac{B_0}{A_0}-\frac{B_1}{A_1}\right)+2\left(\frac{B_1}{A_1}-\frac{B_2}{A_2}\right)+\ldots+2\left(\frac{B_{n-1}}{A_{n-1}}-\frac{B_n}{A_n}\right)$

$=2\left(\frac{B_0}{A_0}-\frac{B_n}{A_n}\right).$

but $B_0=\int_0^{\frac{\pi}{2}}x^2dx=\left[\frac{x^3}{3}\right]_0^{\frac{\pi}{2}}=\frac{\pi^3}{24}, A_0=\int_0^{\frac{\pi}{2}}dx=\frac{\pi}{2}.$

$\therefore \sum_{k=1}^{n}\frac{1}{k^2}=2\left(\frac{\frac{\pi^3}{24}}{\frac{\pi}{2}}-\frac{B_n}{A_n}\right)=\frac{\pi^2}{6}-2\frac{B_n}{A_n}.$

(f) Since $\sin x\ge\frac{2x}{\pi}$, i.e. $\sin^2 x\ge\frac{4x^2}{\pi^2}$, for $0\le x\le\frac{\pi}{2}$,

$B_n\le\int_0^{\frac{\pi}{2}}x^2\left(1-\sin^2 x\right)^n dx$

$\le\int_0^{\frac{\pi}{2}}x^2\left(1-\frac{4x^2}{\pi^2}\right)^n dx.$

(g) Let $u=x, du=dx$,

$dv=x\left(1-\frac{4x^2}{\pi^2}\right)^n dx, v=\frac{-\pi^2}{8(n+1)}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1}.$

$I=\frac{\pi^2}{8(n+1)}\left[x\left(1-\frac{4x^2}{\pi^2}\right)^{n+1}\right]_0^{\frac{\pi}{2}}$

$+\frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1}dx$

$=\frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1}dx.$

(h) Let $x=\frac{\pi}{2}\sin t, dx=\frac{\pi}{2}\cos t\, dt.$

When $x=0, t=0$; when $x=\frac{\pi}{2}, t=\frac{\pi}{2}$

$B_n\le\frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\sin^2 t\right)^{n+1}\cos t\, dt$

$=\frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\cos^{2n+3}t\, dt$

$\le\frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\cos^{2n}t\, dt$, as for $0\le t\le\frac{\pi}{2}, \cos t\le 1$,

the higher the power, the smaller is the value.

$=\frac{\pi^3}{16(n+1)}A_n.$

(i) From (e), $\frac{2B_n}{A_n}=\frac{\pi^2}{6}-\sum_{k=1}^{n}\frac{1}{k^2}$,

from (h), $\frac{2B_n}{A_n}\le\frac{\pi^3}{8(n+1)}.$

$\therefore \frac{\pi^2}{6}-\sum_{k=1}^{n}\frac{1}{k^2}\le\frac{\pi^3}{8(n+1)}.$

$\therefore \frac{\pi^2}{6}-\frac{\pi^3}{8(n+1)}\le\sum_{k=1}^{n}\frac{1}{k^2}.$

From (e), $\sum_{k=1}^{n}\frac{1}{k^2}=\frac{\pi^2}{6}-\frac{2B_n}{A_n}$, and $\frac{2B_n}{A_n}>0$ always,

$\therefore \sum_{k=1}^{n}\frac{1}{k^2}<\frac{\pi^2}{6}.$

$\therefore \frac{\pi^2}{6}-\frac{\pi^3}{8(n+1)}\le\sum_{k=1}^{n}\frac{1}{k^2}<\frac{\pi^2}{6}.$

(j) As $n\to\infty, \frac{\pi^3}{8(n+1)}\to 0, \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2}$ is sandwiched between two values the same, $\therefore \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2}=\frac{\pi^2}{6}.$