Question 1

(a) $\frac{1}{3}\int \frac{3x^2}{(5+x^3)^2}dx = -\frac{1}{3(5+x^3)}+C$.

(b) $\frac{1}{2}\int \frac{2dx}{4x^2+1} = \frac{1}{2}\tan^{-1}(2x)+C$.

(c) $\int \tan^{-1} x\, dx = \left[x\tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}dx$

$$= \left[x\tan^{-1} x - \frac{1}{2}\ln(1+x^2)\right]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

(d) Let $u^2 = 2x-1, 2udu = 2dx, x = \frac{u^2+1}{2}$.

When $x=1, u=1$; when $x=2, u=\sqrt{3}$.

$$\int_1^{\sqrt{3}} \frac{u\,du}{\frac{u^2+1}{2}u} = \int_1^{\sqrt{3}} \frac{du}{u^2+1} = \left[\tan^{-1} u\right]_1^{\sqrt{3}}$$

$$= 2\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \frac{\pi}{6}.$$

(e) $\int_0^1 \frac{4-2x}{2-2x+x^2}dx - \int_0^1 \frac{2x}{2-x^2}dx$

$= \int_0^1 \frac{2-2x}{2-2x+x^2}dx + \int_0^1 \frac{2}{(x-1)^2+1}dx - \int_0^1 \frac{2x}{2-x^2}dx$

$= \left[-\ln(2-2x+x^2) + 2\tan^{-1}(x-1) + \ln(2-x^2)\right]_0^1$

$= \ln\frac{2}{1} + 2\times\frac{\pi}{4} - \ln 2$

$= \frac{\pi}{2}.$

Question 2

(a) $a+ib = 1+6-3i+2i = 7-i$.

$\therefore a = 7, b = -1.$

(b) (i) $\frac{(1+i\sqrt{3})(1-i)}{(1+i)(1-i)} = \frac{(1+\sqrt{3})+i(\sqrt{3}-1)}{2}$.

(ii) $1+i\sqrt{3} = 2\operatorname{cis}\frac{\pi}{3}, 1+i = \sqrt{2}\operatorname{cis}\frac{\pi}{4}$

$$\therefore \frac{1+i\sqrt{3}}{1+i} = \frac{2\operatorname{cis}\frac{\pi}{3}}{\sqrt{2}\operatorname{cis}\frac{\pi}{4}} = \sqrt{2}\operatorname{cis}\frac{\pi}{12}.$$

(iii) $\therefore \sqrt{2}\cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2}$.

$\therefore \cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4}$.

(iii) $\left(\sqrt{2}\operatorname{cis}\frac{\pi}{12}\right)^{12} = 2^6 \operatorname{cis}\pi = -64$.

(c) Let $z = x + iy$,

$z^2 + \bar{z}^2 = x^2 - y^2 + 2ixy + x^2 - y^2 - 2ixy$

$= 2x^2 - 2y^2$.

$\therefore 2x^2 - 2y^2 = 8.$

$x^2 - y^2 = 4.$

$\therefore$ The locus is a rectangular hyperbola.

(d) (i) $M = \frac{\omega z + \bar{\omega} z}{2} = \frac{z}{2}(\omega + \bar{\omega})$

$= \frac{z}{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} + \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)$

$= \frac{z}{2}\times 2\cos\frac{2\pi}{3} = z\cos\frac{2\pi}{3}$

$= -\frac{z}{2}.$

(ii) M is also the midpoint of PS,

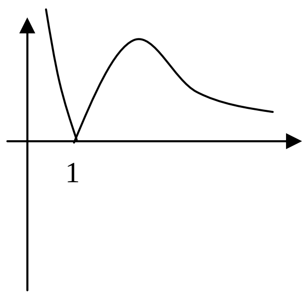
$M = \frac{p+s}{2}$.

$-\frac{z}{2} = \frac{z+s}{2}$.

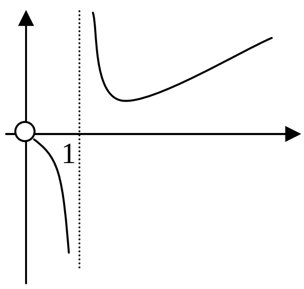
$\therefore s = -2z.$

Question 3

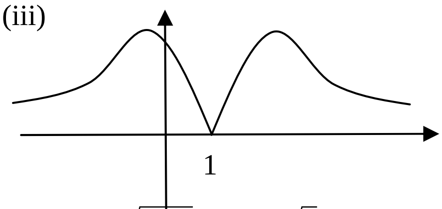
(a) (i)



(ii)



(iii)



(b) (i) $z^2 = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm\sqrt{3}i}{2}$.

As z^2 is not real, z is not real. $\therefore p(z)$ has no real roots.
(ii) $\alpha^6 - 1 = (\alpha^2)^3 - 1 = (\alpha^2 - 1)(\alpha^4 + \alpha^2 + 1)$.
$\therefore$ If α is a root of $1 + z^2 + z^4 = 0$, it satisfies $z^6 - 1 = 0$.
$\therefore \alpha^6 = 1$.
(iii) Substitute z by α^2,
$1 + \alpha^4 + \alpha^8 = 1 + \alpha^4 + \alpha^6.\alpha^2$
$= 1 + \alpha^4 + \alpha^2 = 0$
$\therefore \alpha^2$ is a zero of $p(z)$.

(c) (i) $I_n + I_{n-1} = \int_0^{\frac{\pi}{4}} (\tan^{2n}\theta + \tan^{2(n-1)}\theta)d\theta$

$= \int_0^{\frac{\pi}{4}} \tan^{2(n-1)}\theta(\tan^2\theta + 1)d\theta$

$= \int_0^{\frac{\pi}{4}} \tan^{2(n-1)}\theta\sec^2\theta d\theta$

$= \left[\frac{\tan^{2n-1}\theta}{2n-1}\right]_0^{\frac{\pi}{4}} = \frac{1}{2n-1}.$

(ii) $I_3 + I_2 = \frac{1}{5}, I_2 + I_1 = \frac{1}{3}, I_1 + I_0 = 1$

$I_0 = \int_0^{\frac{\pi}{4}} 1 d\theta = \frac{\pi}{4}.$

$\therefore I_3 = \frac{1}{5} - \left[\frac{1}{3} - \left(1 - \frac{\pi}{4}\right)\right] = \frac{13}{15} - \frac{\pi}{4}.$

(d) Resolving the forces
Vertically, $mg = T\cos\alpha$. (1)
Horizontally, $T\sin\alpha = mr\omega^2 = m\ell\sin\alpha\omega^2$.
$m\ell\omega^2 = T$ (2)
$\frac{(2)}{(1)}$ gives $\frac{m\ell\omega^2}{mg} = \frac{T}{T\cos\alpha}$.

$\therefore \omega^2 = \frac{g}{\ell\cos\alpha}$.

Question 4

(a) (i) $\frac{1}{2}rk$.
(ii) $\Delta KLM = \Delta OLM + \Delta OKL + \Delta OKM$
$= \frac{1}{2}rk + \frac{1}{2}rm + \frac{1}{2}r\ell$
$= \frac{1}{2}r(k + m + \ell)$
$= \frac{1}{2}rP$.
(iii) Let the distance from where the wheel touches the ground to where the board touches the ground be k.

Using the result of part (ii)
$\frac{1}{2}\times 8\times(2+k) = \frac{1}{2}\times 2\times(6+6+2k+4)$
$16 + 8k = 32 + 4k$.
$4k = 16$.
$k = 4$.
$\therefore$ The board touches the ground at a point 6 m from the fence.
(iv) The lengths of the boards, let them be s and t, can be found by Pythagoras' theorem:
$s = \sqrt{8^2 + 6^2} = 10, t = \sqrt{8^2 + (6+9)^2} = 17$.
Using the same result of part (ii),
$\frac{1}{2}\times 9\times 8 = \frac{1}{2}\times r\times(9+10+17)$
$\therefore r = \frac{72}{36} = 2$ units.

(b) (i) $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0.$
$\frac{dy}{dx} = -\frac{b^2x}{a^2y}.$
At $P(x_1, y_1), m = -\frac{b^2x_1}{a^2y_1}$.
Eqn of the tangent:
$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$
$a^2y_1y - a^2y_1^2 = -b^2x_1x + b^2x_1^2$.
$b^2x_1x + a^2y_1y = a^2y_1^2 + b^2x_1^2$.
$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$
$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$ (since $(x_1, y_1) \in$ ellipse) (1)

(ii) Similarly, the tangent at Q is $\frac{x_2x}{a^2} + \frac{y_2y}{b^2} = 1$. (2)

(1) − (2) gives $\frac{(x_1 - x_2)}{a^2}x + \frac{(y_1 - y_2)}{b^2}y = 0$. (3)

Since T is the point of intersection of the two lines, T satisfies the equation (3) above.
(iii) Since O (0,0) also satisfies the equation (3), this equation is the equation of OT.
Substituting $M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ to (3),

$\text{LHS} = \frac{(x_1 - x_2)(x_1 + x_2)}{2a^2} + \frac{(y_1 - y_2)(y_1 + y_2)}{2b^2}$
$= \frac{x_1^2 - x_2^2}{2a^2} + \frac{y_1^2 - y_2^2}{2b^2}$

$=\frac{1}{2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)-\frac{1}{2}\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}\right)$

$=\frac{1}{2}-\frac{1}{2}=0=\text{RHS}.$

$\therefore O, T, M$ are collinear.

Question 5

(a) (i) $\frac{dP}{dt}=21000\left(\frac{e^{-\frac{t}{3}}}{(7+3e^{-\frac{t}{3}})^2}\right)=\frac{21000}{7+3e^{-\frac{t}{3}}}\frac{e^{-\frac{t}{3}}}{7+3e^{-\frac{t}{3}}}$

but $1-\frac{P}{3000}=1-\frac{21}{7+3e^{-\frac{t}{3}}}=\frac{3e^{-\frac{t}{3}}}{7+3e^{-\frac{t}{3}}}$,

$\therefore \frac{dP}{dt}=\frac{1}{3}\left(1-\frac{P}{3000}\right)P.$

(ii) When $t = 0$, $P=\frac{21000}{7+3}=2100$.

(iii) When $t\to\infty, e^{-\frac{t}{3}}\to 0, P=\frac{21000}{7}=3000$.

(iv) When $t = 0$, $\frac{dP}{dt}=\frac{1}{3}\left(1-\frac{2100}{3000}\right)2100=210$.

$\therefore \frac{210}{2100}=0.1 = 10\%.$

(b) (i) $\frac{d}{dx}p(x)=(n+1)x^n-(n+1)$

$=0$ when $x^n=1, \therefore$ One root is $x=1$.

$p(1)=1-(n+1)+n=0.$

As 1 satisfies $p(x)=0$ and $\frac{d}{dx}p(x)=0$, 1 is the double root.

(ii) $\frac{d^2}{dx^2}p(x)=n(n+1)x^{n-1}.$

For $x>0, \frac{d^2}{dx^2}p(x)>0$: The curve is concave up for $x\geq 0$.

The curve has a double root at $x = 1$, i.e. it touches the x-axis at $x = 1$, and for $x \geq 0$, it is concave up, $\therefore\ p(x)\geq 0$ for all $x \geq 0$.

(iii) When $n = 3$, $p(x)=x^4-4x+3$.

Since $(x-1)^2=x^2-2x+1$,

$x^4-4x+3=(x^2-2x+1)(x^2+3x+3)$

$=(x-1)^2(x^2+3x+3).$

(c) (i) $x-a=\pm\sqrt{b^2-h^2}$.

$\therefore x_1=a-\sqrt{b^2-h^2}, x_2=a+\sqrt{b^2-h^2}.$

(ii) Area = $\pi x_2^2-\pi x_1^2$

$=\pi\left(a^2+(b^2-h^2)+2a\sqrt{b^2-h^2}\right)$

$-\pi\left(a^2+(b^2-h^2)+2a\sqrt{b^2-h^2}\right)$

$=4\pi a\sqrt{b^2-h^2}.$

(iii) $\partial V=4\pi a\sqrt{b^2-h^2}\partial h.$

$V=4\pi a\int_{-a}^{a}\sqrt{b^2-h^2}dh=4\pi a\times\frac{1}{2}\pi b^2=2\pi^2ab^2.$

Question 6

(a) $w=\sqrt[3]{1}=\sqrt[3]{\text{cis}(2k\pi)}=\text{cis}\frac{2k\pi}{3}=1, \text{cis}\frac{2\pi}{3}, \text{cis}\left(\frac{-2\pi}{3}\right).$

Let $w=\text{cis}\frac{2\pi}{3}, \overline{w}=\text{cis}\left(\frac{-2\pi}{3}\right)=\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}$,

$\therefore w+\overline{w}=2\cos\frac{2\pi}{3}=-1, w\overline{w}=\text{cis}\frac{2\pi}{3}\text{cis}\left(\frac{-2\pi}{3}\right)=1.$

$p(z)=(z-1)(z+w)(z+\overline{w})$

$=(z-1)(z^2+(w+\overline{w})z+w\overline{w})$

$=(z-1)(z^2-z+1)$

$=z^3-2z^2+2z-1.$

(b) (i) $m=\frac{dy/d\theta}{dx/d\theta}=\frac{b\sec^2\theta}{a\sec\theta\tan\theta}=\frac{b\sec\theta}{a\tan\theta}.$

$y-b\tan\theta=\frac{b\sec\theta}{a\tan\theta}(x-a\sec\theta).$

$ay\tan\theta-ab\tan^2\theta=bx\sec\theta-ab\sec^2\theta.$

$bx\sec\theta-ay\tan\theta=ab(\sec^2\theta-\tan^2\theta)$

$=ab.$

(ii) $SR=\frac{|bae\sec\theta-0-ab|}{\sqrt{b^2\sec^2\theta+a^2\tan^2\theta}}$

$=\frac{|ab(e\sec\theta-1)|}{\sqrt{b^2\sec^2\theta+a^2\tan^2\theta}}.$

(iii) Similarly, $S'R'=\frac{|ab(e\sec\theta+1)|}{\sqrt{b^2\sec^2\theta+a^2\tan^2\theta}}.$

$SR\times S'R'=\frac{|ab(e\sec\theta-1)|}{\sqrt{b^2\sec^2\theta+a^2\tan^2\theta}}$

$\times\frac{|ab(e\sec\theta+1)|}{\sqrt{b^2\sec^2\theta+a^2\tan^2\theta}}$

$=\frac{a^2b^2(e^2\sec^2\theta-1)}{b^2\sec^2\theta+a^2\tan^2\theta}$

$=\frac{a^2b^2(e^2\sec^2\theta-1)}{a^2(e^2-1)\sec^2\theta+a^2\tan^2\theta}$

$$=\frac{b^2(e^2\sec^2\theta-1)}{e^2\sec^2\theta-\sec^2\theta+\tan^2\theta}$$
$$=\frac{b^2(e^2\sec^2\theta-1)}{e^2\sec^2\theta-1}$$
$$=b^2.$$

(c) (i) LHS = $\dfrac{1}{\dfrac{n!}{r!(n-r)!}}=\dfrac{r!(n-r)!}{n!}$

$$\text{RHS}=\frac{r}{r-1}\left[\frac{(r-1)!(n-r)!}{(n-1)!}-\frac{(r-1)!(n-r+1)!}{n!}\right]$$
$$=\frac{r}{r-1}\left[\frac{(r-1)!(n-r)!}{n!}\big(n-(n-r+1)\big)\right]$$
$$=\frac{r}{r-1}\left[\frac{(r-1)!(n-r)!}{n!}(r-1)\right]$$
$$=\frac{r!(n-r)!}{n!}=\text{LHS}.$$

(ii) LHS = $\dfrac{r}{r-1}\left[\dfrac{1}{\binom{r-1}{r-1}}-\dfrac{1}{\binom{r}{r-1}}+\dfrac{1}{\binom{r}{r-1}}-\dfrac{1}{\binom{r+1}{r-1}}\right.$

$$\left.+\ldots+\frac{1}{\binom{m-2}{r-1}}-\frac{1}{\binom{m-1}{r-1}}+\frac{1}{\binom{m-1}{r-1}}-\frac{1}{\binom{m}{r-1}}\right]$$

$$=\frac{r}{r-1}\left[\frac{1}{1}-\frac{1}{\binom{m}{r-1}}\right].$$

(iii) When $m\to\infty,\binom{m}{r-1}\to\infty,\therefore\sum_{n=r}^{m}\dfrac{1}{\binom{n}{r}}\to\dfrac{r}{r-1}$.

Question 7

(a) (i) $p_s=\dfrac{\binom{3}{1}\binom{n}{3}}{\binom{3n}{3}}$, (ii) $p_d=\dfrac{\binom{n}{1}^3}{\binom{3n}{3}}$,

(iii) $p_m=\dfrac{\binom{3}{1}\binom{n}{1}\binom{2}{1}\binom{n}{2}}{\binom{3n}{3}}$,

(iv) $\binom{3}{1}\binom{n}{3}:\binom{n}{1}^3:\binom{3}{1}\binom{n}{2}\binom{2}{1}\binom{n}{1}$

$$=3\frac{n(n-1)(n-2)}{3!}:n^3:6\frac{n(n-1)}{2}n. \qquad (1)$$

When n is large, $\dfrac{n(n-1)(n-2)}{3!}\simeq\dfrac{n^3}{6}$,

and $\dfrac{n(n-1)n}{2}\simeq\dfrac{n^3}{2}$.

(1) becomes $\dfrac{1}{2}:1:3$, which is $1:2:6$.

(b) (i) $\angle TSP$ = $\angle SQP$ + $\angle SPT$ (exterior angle = sum of two opposite interior angles)
But $\angle SQP$ = $\angle RPT$ (angles in alternate segments are equal) and $\angle SPT$ = $\angle SPR$ (given).
$\therefore\angle TSP$ = $\angle RPT$ + $\angle SPR$ = $\angle TPS$.
(ii) $TR\times TQ=TP^2$ (If a tangent and a secant intersect, the product of the secant and its external segment equals the square of the tangent)

$(c-a)(c-a+a+b)=c^2.$

$(c-a)(c+b)=c^2.$

$c^2+cb-ac-ab=c^2.$

$cb=ac+ab.$

$\dfrac{1}{a}=\dfrac{1}{b}+\dfrac{1}{c}$, on dividing by abc.

(c) (i) $\dfrac{dv}{dt}=-(b-v_0)\times(-\alpha)e^{-\alpha t}=\alpha(b-v_0)e^{-\alpha t}$

$=\alpha(b-v).$

(ii) b is the current's speed.

(iii) $v=\dfrac{dx}{dt}=b-(b-v_0)e^{-\alpha t}$

$$x=bt+\frac{b-v_0}{\alpha}e^{-\alpha t}+C.$$

When $t=0,v=v_0,x=0\ \therefore C=-\dfrac{b-v_0}{\alpha}$.

$$\therefore x=bt+\frac{b-v_0}{\alpha}e^{-\alpha t}-\frac{b-v_0}{\alpha}.$$

But $e^{-\alpha t}=\dfrac{b-v}{b-v_0},\therefore t=-\dfrac{1}{\alpha}\ln\dfrac{b-v}{b-v_0}$.

$$\therefore x=-\frac{b}{\alpha}\ln\frac{b-v}{b-v_0}+\frac{b-v_0}{\alpha}\frac{b-v}{b-v_0}-\frac{b-v_0}{\alpha}$$
$$=\frac{b}{\alpha}\ln\frac{b-v_0}{b-v}+\frac{b-v-b+v_0}{\alpha}$$
$$=\frac{b}{\alpha}\ln\frac{b-v_0}{b-v}+\frac{v_0-v}{\alpha}.$$

(iv) $\frac{b}{\alpha}\ln\frac{b-0.1b}{b-0.5b}+\frac{0.1b-0.5b}{\alpha}=\frac{b}{\alpha}\ln\frac{9}{5}-0.4\frac{b}{\alpha}=0.18\frac{b}{\alpha}$.

$\therefore$ It has drifted $0.18\frac{b}{\alpha}$ units.

Question 8

(a) When $n = 1$, LHS = $\cos\theta$, RHS = $\frac{\sin 2\theta}{2\sin\theta}=\frac{2\sin\theta\cos\theta}{2\sin\theta}=\cos\theta$.

Assume $\cos\theta+\cos 3\theta+...+\cos(2k-1)\theta=\frac{\sin 2k\theta}{2\sin\theta}$.

$\cos\theta+\cos 3\theta+...+\cos(2k-1)\theta+\cos(2k+1)\theta$

$=\frac{\sin 2k\theta}{2\sin\theta}+\cos(2k+1)\theta$

$=\frac{\sin 2k\theta+2\cos(2k+1)\theta\sin\theta}{2\sin\theta}$

$=\frac{\sin 2k\theta+2\cos 2k\theta\cos\theta\sin\theta-2\sin 2k\theta\sin^2\theta}{2\sin\theta}$

$=\frac{\cos 2k\theta\sin 2\theta+\sin 2k\theta(1-2\sin^2\theta)}{2\sin\theta}$

$=\frac{\cos 2k\theta\sin 2\theta+\sin 2k\theta\cos 2\theta}{2\sin\theta}$

$=\frac{\sin(2k+2)\theta}{2\sin\theta}=\frac{\sin 2(k+1)\theta}{2\sin\theta}$.

$\therefore$ The statement is true for $n = k + 1$.

$\therefore$ It is true for all $n \geq 1$.

(b) (i) $A=2\pi R^2\sin\delta\left(\cos\frac{\delta}{2}+\cos\frac{3\delta}{2}+...+\cos\frac{(2k-1)\delta}{2}\right)$

$=2\pi R^2\sin\delta\times\frac{\sin n\delta}{2\sin\frac{\delta}{2}}$

$=2\pi R^2\times 2\sin\frac{\delta}{2}\cos\frac{\delta}{2}\times\frac{\sin n\delta}{2\sin\frac{\delta}{2}}$

$=2\pi R^2\cos\frac{\delta}{2}\sin n\delta$

$=2\pi R^2\cos\frac{\pi}{4n}$, since $\frac{\pi}{2}=n\delta, \therefore \sin n\delta=1$.

(ii) As $n\to\infty, \frac{\pi}{4n}\to 0, \cos\frac{\pi}{4n}\to 1, \therefore A\to 2\pi R^2$.

(c) (i) $f'(t)=n\cos(a+nt)\sin b-(-n)\sin a\cos(b-nt)$.

$=n\cos(a+nt)\sin b+n\sin a\cos(b-nt)$.

$f''(t)=-n^2\sin(a+nt)\sin b+n(-n)\sin a(-\sin(b-nt))$

$=-n^2\sin(a+nt)\sin b+n^2\sin a\sin(b-nt)$

$=-n^2\left(\sin(a+nt)\sin b-\sin a\sin(b-nt)\right)$

$=-n^2 f(t)$.

Also, $f(0)=\sin a\sin b-\sin a\sin b=0$.

(ii) $f(t)=\sin a\sin b\cos nt+\cos a\sin b\sin nt$

$-\sin a\sin b\cos nt+\sin a\cos b\sin nt$

$=\cos a\sin b\sin nt+\sin a\cos b\sin nt$

$=\left(\sin a\cos b+\cos a\sin b\right)\sin nt$

$=\sin(a+b)\sin nt$.

Note: It can be proven using the result of (i) that $f(t)$ is simple harmonic motion.

(iii) $\frac{\sin(a+nt)}{\sin(b-nt)}=\frac{\sin a}{\sin b}$ occurs when $f(t) = 0$.

Solving $\sin(a+b)\sin nt=0$ gives $nt=k\pi$,

$\therefore t=\frac{k\pi}{n}, k\in J$.