HSC 2005
Mathematics Extension 2 Solution

Question 1

(a) Let $u=\sin\theta, du=\cos\theta\,d\theta; \int \frac{du}{u^5} = -\frac{1}{4u^4} = -\frac{1}{4\sin^4\theta} + C$.

Alternatively, $\int \frac{\cos\theta}{\sin^5\theta}d\theta = \int \cot\theta \operatorname{cosec}^4\theta d\theta = \int \operatorname{cosec}^3\theta \times \cot\theta \operatorname{cosec}\theta d\theta = -\frac{\operatorname{cosec}^4\theta}{4} + C.$

(b) (i) $a = \lim_{x\to 3}\frac{5x}{x+2} = 3, b = \lim_{x\to -2}\frac{5x}{x-3} = 2$,

(ii) $\int \frac{5x\,dx}{(x-3)(x+2)} = 3\ln(x-3) + 2\ln(x+2) + C$.

(c) Let $u=\ln x, du = \frac{1}{x}dx$; Let $dv = x^7 dx, v = \frac{x^8}{8}$.

$\therefore \int_1^e x^7 \ln x\,dx = \left[\frac{x^8\ln x}{8}\right]_1^e - \int_1^e \frac{x^7}{8}dx = \left[\frac{x^8\ln x}{8} - \frac{x^8}{64}\right]_1^e = \left(\frac{e^8}{8} - \frac{e^8}{64}\right) - \left(-\frac{1}{64}\right) = \frac{7e^8+1}{64}$.

(d) $\int \frac{dx}{\sqrt{4x^2-1}} = \frac{1}{2}\ln\left(2x + \sqrt{4x^2-1}\right) + C$.

(e) (i) $\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2} = \frac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right) = \frac{1}{2}\left(1+t^2\right)$.

(ii) $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}} = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{2t}{1+t^2}$.

(iii) $\int \operatorname{cosec}\theta\,d\theta = \int \frac{d\theta}{\sin\theta} = \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}} = \int \frac{dt}{t} = \ln t = \ln\left(\tan\frac{\theta}{2}\right) + C.$

Question 2

(a) (i) $2z + iw = 6 + 2i + i(1-i) = 7 + 3i$.

(ii) $\bar{z}w = (3-i)(1-i) = 2-4i$.

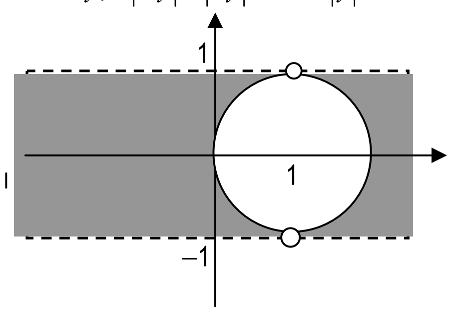
(iii) $\frac{6}{w} = \frac{6}{1-i} = \frac{6(1+i)}{2} = 3(1+i)$.

(b) (i) $\beta = 2\operatorname{cis}\left(-\frac{\pi}{3}\right)$.

(ii) $\beta^5 = 2^5\operatorname{cis}\left(-\frac{5\pi}{3}\right) = 32\operatorname{cis}\frac{\pi}{3}$.

(iii) $\beta^5 = 32\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 16\left(1+i\sqrt{3}\right)$.

(c) Let $z = x+iy$, so $\bar{z} = x - iy, \therefore z - \bar{z} = 2iy; \therefore |2iy| = |2y| < 2 \Leftrightarrow |y| < 1 \Leftrightarrow -1 < y < 1$.



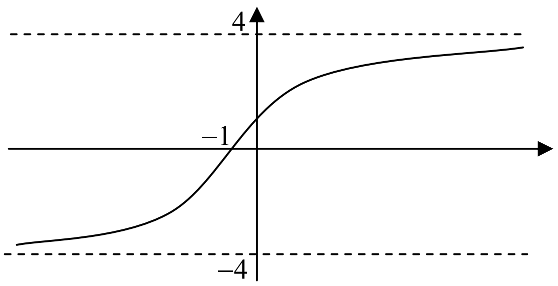
(d) (i) Let $\angle POQ = 2\theta$, then $\arg(z_1)+\arg(z_2)=(\alpha-\theta)+(\alpha+\theta)=2\alpha$.

(ii) $z_1 = |z_1|\text{cis}(\alpha-\theta), z_2 = |z_1|\text{cis}(\alpha+\theta), \therefore z_1 z_2 = |z_1|^2 \text{cis}(\alpha-\theta+\alpha+\theta) = |z_1|^2 \text{cis}(2\alpha)$.

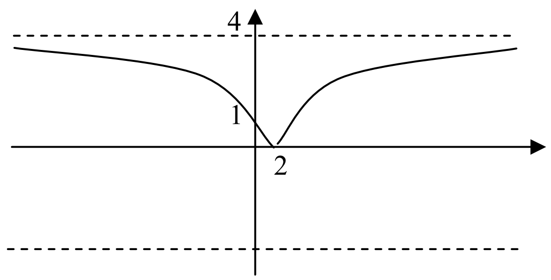
(iii) $R = |z_1|^2 \text{cis}(2\alpha) = |z_1|^2 \text{cis}\frac{\pi}{2} = i|z_1|^2$, purely imaginary and $|z_1|^2 > 0$, so the locus of R is the y-axis, $y > 0$.

Question 3

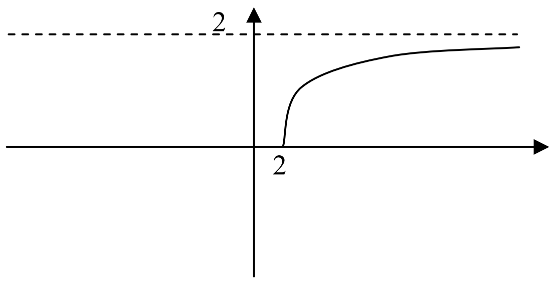
(a) (i)



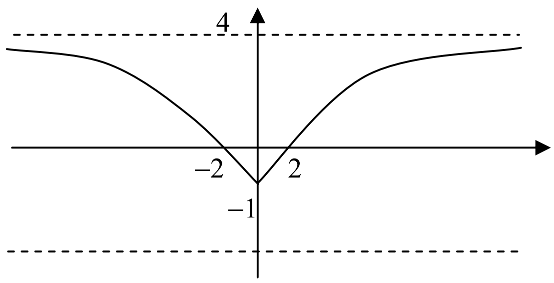
(ii)



(iii)



(iv)

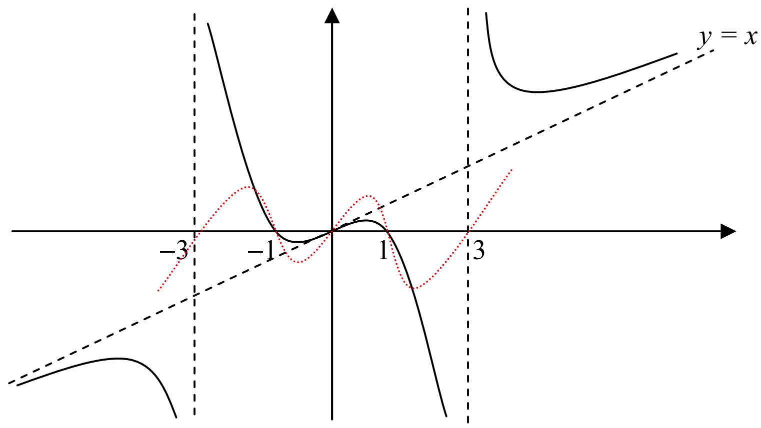


(b) $y = x + \frac{8x}{x^2-9} = \frac{x^3-x}{x^2-9} = \frac{x(x+1)(x-1)}{(x+3)(x-3)}$

$\therefore$ Asymptotes: $y = x$, and $x = \pm 3$

$\therefore x$-intercepts: (0,0), (±1,0).

(The dotted curve is the "guidegraph" of equation $y=(x+3)(x+1)x(x-1)(x-3)$)



(c) Differentiate implicitly, $3x^2 - 4y - 4xy' + 3y^2 y' = 0$.

$3x^2 - 4y - (4x - 3y^2)y' = 0$

$y' = \frac{3x^2-4y}{4x-3y^2}$.

At $(2,1), m_1 = \frac{12-4}{8-3} = \frac{8}{5}; \therefore m_2 = -\frac{5}{8}$.

The equation of the normal is: $y - 1 = -\frac{5}{8}(x-2)$

$8y - 8 = -5x + 10$

$5x + 8y - 18 = 0$.

(d) Resolving the forces

Vertically, $N\cos\theta = mg$. (1)

Horizontally, $N\sin\theta = \frac{mv^2}{r}$. (2)

$(1)^2 + (2)^2$ gives $N^2 = m^2\left(g^2 + \frac{v^4}{r^2}\right)$.

$\therefore N = m\sqrt{g^2 + \frac{v^4}{r^2}}$.

Question 4

(a) (i) Consider a strip of thickness δx at the distance of x from the origin. As the strip is rotated about the y-axis the volume of the generated hollow cylinder is

$$\delta V = \pi\left((x+\delta x)^2 - x^2\right)y$$
$$\simeq 2\pi xy\delta x.$$

$$V = 2\pi\int_0^N xy\,dx = 2\pi\int_0^N xe^{-x^2}dx = 2\pi\left[-\frac{1}{2}e^{-x^2}\right]_0^N = \pi\left(1 - e^{-N^2}\right).$$

(ii) As $N \to \infty, e^{-N^2} \to 0, V \to \pi$.

(b) (i) $\alpha + \beta + \gamma + \delta = -p$.

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r$.

(ii) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2\sum\alpha\beta = (-p)^2 - 2q = p^2 - 2q$.

(iii) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 9 - 2(5) = -1 < 0$, so it cannot have four real roots.

(iv) When $x = 0, p(0) = -8,$ and when $x = 1, p(1) = 2$, $\therefore$ it has at least a real root between 0 and 1. But from (iii) it cannot have four real roots, and as the coefficients are real, it must have a pair of conjugate roots, and two real roots.

(c) (i) Put $(0, -b)$ into the equation of the normal:

$$b^3x_1 = (a^2 - b^2)x_1y_1$$
$$b^3 = (a^2 - b^2)y_1$$
$$y_1 = \frac{b^3}{a^2 - b^2}.$$

Also, from the diagram, the normal to the ellipse at $(0, \pm b)$ is the y-axis. $\therefore$ This normal passes through $y_1 = \pm b$.

(ii) If $\frac{b^3}{a^2 - b^2} \le b$ then $b^2 < a^2 - b^2$. $\therefore 2b^2 \le a^2$

$$\frac{b^2}{a^2} \le \frac{1}{2}$$

But $b^2 = a^2(1 - e^2), \therefore 1 - e^2 = \frac{b^2}{a^2}, \therefore e^2 = 1 - \frac{b^2}{a^2} \ge 1 - \frac{1}{2} = \frac{1}{2}, \therefore e \ge \frac{1}{\sqrt{2}}$ (Don't forget that $e < 1$)

Question 5

(a) (i) Area of $\Delta\, ABC = \frac{1}{2}bc = \frac{1}{2}ad$

$$\therefore bc = ad$$
$$b^2c^2 = a^2d^2$$
$$= d^2(b^2 + c^2).$$

(ii) $\tan\alpha = \frac{h}{AB}, \therefore AB = \frac{h}{\tan\alpha}$.

Similarly, $AP = \frac{h}{\tan\gamma}, AC = \frac{h}{\tan\beta}$.

From part (i), $AB^2.AC^2 = AP^2\left(AB^2 + AC^2\right)$,

$$\frac{h^2}{\tan^2\alpha}.\frac{h^2}{\tan^2\beta} = \frac{h^2}{\tan^2\gamma}\left(\frac{h^2}{\tan^2\alpha} + \frac{h^2}{\tan^2\beta}\right)$$

$$\tan^2\gamma = \left(\frac{1}{\tan^2\alpha} + \frac{1}{\tan^2\beta}\right)\tan^2\alpha.\tan^2\beta$$

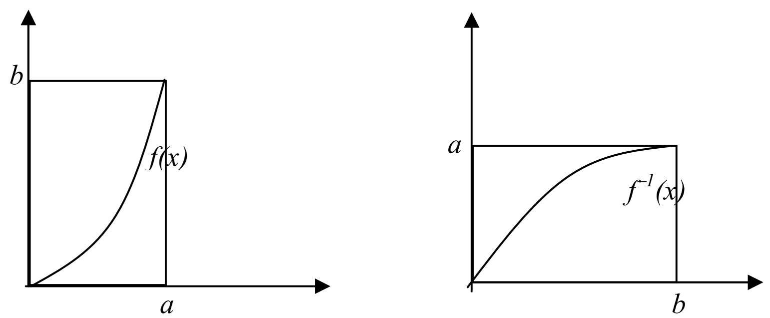
$$= \tan^2\beta + \tan^2\alpha.$$

(b) (i) If Ferdinand only scores 1 goal, they must play 6 goals, of which Mary won the last goal. Therefore, Ferdinand can win any of the previous five goals.

(ii) Assume the winner of the last goal is Mary, the game can be won by one of the following ways (5M), (1F, 4M), (2F, 4M), (3F, 4M), (4F, 4M). Therefore, the total number of ways is 1 + $\frac{5!}{4!} + \frac{6!}{2!4!} + \frac{7!}{3!4!} + \frac{8!}{4!4!} = 126$.

Of course, Ferdinand can win by the same way, so the total outcome is $126 \times 2 = 252$ ways.

(c) (i)



From the diagram, it's clear that the area under the curve of $f(x)$ + the area under the curve of $f^{-1}(x)$ is the area of the rectangle.

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx = ab$$

$$\therefore \int_0^a f(x)dx = ab - \int_0^b f^{-1}(x)dx.$$

(ii) When $x = 2, \sin^{-1}\left(\frac{2}{4}\right) = \frac{\pi}{6}$.

Also, if $f: y = \sin^{-1}\frac{x}{4}$ then $f^{-1}: x = \sin^{-1}\frac{y}{4}, \therefore y = 4\sin x$.

$$\int_0^2 \sin^{-1}\left(\frac{x}{4}\right) = 2\times\frac{\pi}{6} - \int_0^{\frac{\pi}{6}} 4\sin x\, dx$$

$$= \frac{\pi}{3} + 4\Big[\cos x\Big]_0^{\frac{\pi}{6}}$$

$$= \frac{\pi}{3} + 4\left(\frac{\sqrt{3}}{2} - 1\right)$$

$$= \frac{\pi}{3} + 2\sqrt{3} - 4.$$

(d) (i) Area of $ABCD = AD \times CD$, where $AD = 2\sqrt{9-x^2}$, by Pythagoras, and $CD = x \times \tan 60° = \sqrt{3}x$.

$\therefore$ Area of $ABCD = 2\sqrt{3}x\sqrt{9-x^2} = 2x\sqrt{27-3x^2}$.

(ii) $V = \int_0^3 2x\sqrt{27-3x^2}\,dx = -\frac{1}{3}\left[\frac{2(27-3x^2)^{3/2}}{3}\right]_0^3 = -\frac{1}{3}\left(0-\frac{2(27)^{3/2}}{3}\right) = 18\sqrt{3}$.

Question 6

(a) (i) Let $n = 0, I_0(x) = \int_0^x e^{-t}dt = \left[-e^{-t}\right]_0^x = 1-e^{-x}$. $\therefore$ True for $n = 0$.

Assume $I_n(x) = n!\left[1-e^{-x}\left(1+x+\frac{x^2}{2!}+...+\frac{x^n}{n!}\right)\right]$.

Required to prove that $I_{n+1}(x) = (n+1)!\left[1-e^{-x}\left(1+x+\frac{x^2}{2!}+...+\frac{x^n}{n!}+\frac{x^{n+1}}{(n+1)!}\right)\right]$.

LHS = $\int_0^x t^{n+1}e^{-t}dt$.

Using Integration by parts, let $u = t^{n+1}, du = (n+1)t^n$. Let $dv = e^{-t}dt, v = -e^{-t}$.

$$\begin{aligned}\int_0^x t^{n+1}e^{-t}dt &= \left[-t^{n+1}e^{-t}\right]_0^x + (n+1)\int_0^x t^n e^{-t}dt \\ &= -x^{n+1}e^{-x} + (n+1)I_n(x) \\ &= -x^{n+1}e^{-x} + (n+1)n!\left[1-e^{-x}\left(1+x+\frac{x^2}{2!}+...+\frac{x^n}{n!}\right)\right] \\ &= -x^{n+1}e^{-x} + (n+1)!\left[1-e^{-x}\left(1+x+\frac{x^2}{2!}+...+\frac{x^n}{n!}\right)\right] \\ &= (n+1)!\left[1-e^{-x}\left(1+x+\frac{x^2}{2!}+...+\frac{x^n}{n!}+\frac{x^{n+1}}{(n+1)!}\right)\right] = \text{RHS}.\end{aligned}$$

$\therefore$ It's true for all $n \geq 0$.

(ii) For $0 < t < 1, t^n e^{-t} > 0, \therefore \int_0^1 t^n e^{-t}dt > 0$.

For $0 < t < 1, e^{-t} < 1, \therefore \int_0^1 t^n e^{-t}dt < \int_0^1 t^n dt = \left[\frac{t^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}$.

(iii) $0 \leq n!\left[1-e^{-x}\left(1+x+\frac{x^2}{2!}+...+\frac{x^n}{n!}\right)\right]_0^1 \leq \frac{1}{(n+1)}$.

$0 \leq \left[1-e^{-x}\left(1+x+\frac{x^2}{2!}+...+\frac{x^n}{n!}\right)\right]_0^1 \leq \frac{1}{n!(n+1)} = \frac{1}{(n+1)!}$.

$\therefore 0 \leq 1-e^{-1}\left(1+\frac{1}{1!}+\frac{1}{2!}+...+\frac{1}{n!}\right) \leq \frac{1}{(n+1)!}$.

(iv) As $n \to \infty, \frac{1}{(n+1)!} \to 0, \therefore$ as $1-e^{-1}\left(1+\frac{1}{1!}+\frac{1}{2!}+...+\frac{1}{n!}\right)$, where $n \to \infty$, is sandwiched between two values which are the same, $1-e^{-1}\left(1+\frac{1}{1!}+\frac{1}{2!}+...\right) = 0$.

$$1 = e^{-1}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\right)$$

$$\therefore 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots = \frac{1}{e^{-1}} = e.$$

(b) (i) Given $\omega^n = 1, \therefore 1 - \omega^n = 0$. $\therefore (1-\omega)(1+\omega+\omega^2+\ldots+\omega^{n-1}) = 0$. $\therefore 1+\omega+\omega^2+\ldots+\omega^{n-1} = 0.$

$$\left(1+2\omega+3\omega^2+\ldots+n\omega^{n-1}\right)(\omega-1) = \omega - 1 + 2\omega^2 - 2\omega + 3\omega^3 - 3\omega^2 + \ldots + n\omega^n - n\omega^{n-1}$$

$$= -(1+\omega+\omega^2+\ldots+\omega^{n-1}) + n\omega^n = 0 + n\omega^n = n, \text{ since } \omega^n = 1.$$

(ii) $\dfrac{1}{\cos 2\theta + i\sin 2\theta - 1} = \dfrac{1}{z^2 - 1}$, if $z = \cos\theta + i\sin\theta$, so $z^2 = \cos 2\theta + i\sin 2\theta$,

$$= \frac{z^{-1}}{z - z^{-1}}$$

$$= \frac{\cos\theta - i\sin\theta}{(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)}, \text{ since } (\cos\theta + i\sin\theta)^{-1} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$$

$$= \frac{\cos\theta - i\sin\theta}{2i\sin\theta}.$$

(iii) If $\omega = \text{cis}\dfrac{2\pi}{n}$ then $\omega = \alpha^2$, where $\alpha = \text{cis}\dfrac{\pi}{n}$, $\therefore \text{Re}\left(\dfrac{1}{\omega-1}\right) = \text{Re}\left(\dfrac{1}{\alpha^2-1}\right) = \text{Re}\left(\dfrac{\cos\theta - i\sin\theta}{2i\sin\theta}\right)$, where $\theta = \dfrac{\pi}{n}$.

But $\text{Re}\left(\dfrac{\cos\theta - i\sin\theta}{2i\sin\theta}\right) = -\dfrac{1}{2}$, for all θ, $\therefore \text{Re}\left(\dfrac{1}{\omega-1}\right) = -\dfrac{1}{2}$.

(iv) $1 + 2\cos\dfrac{2\pi}{5} + 3\cos\dfrac{4\pi}{5} + 4\cos\dfrac{6\pi}{5} + 5\cos\dfrac{8\pi}{5} = \text{Re}\left(1 + 2\omega + 3\omega^2 + 4\omega^3 + 5\omega^4\right)$

$$= \text{Re}\left(\frac{5}{\omega-1}\right), \text{ from (i)}$$

$$= 5\left(-\frac{1}{2}\right) = -\frac{5}{2}.$$

(v) $1 + 2\cos\dfrac{2\pi}{5} + 3\cos\dfrac{4\pi}{5} + 4\cos\dfrac{6\pi}{5} + 5\cos\dfrac{8\pi}{5} = 1 + 2\cos\dfrac{2\pi}{5} - 3\cos\dfrac{\pi}{5} - 4\cos\dfrac{\pi}{5} + 5\cos\dfrac{2\pi}{5} = 1 + 7\cos\dfrac{2\pi}{5} - 7\cos\dfrac{\pi}{5}$.

$$\therefore 7\cos\frac{2\pi}{5} - 7\cos\frac{\pi}{5} = -\frac{5}{2} - 1 = -\frac{7}{2}$$

$$\therefore \cos\frac{2\pi}{5} - \cos\frac{\pi}{5} = -\frac{1}{2}.$$

$$2\cos^2\frac{\pi}{5} - 1 - \cos\frac{\pi}{5} = -\frac{1}{2}.$$

$$2\cos^2\frac{\pi}{5} - \cos\frac{\pi}{5} - \frac{1}{2} = 0.$$

$$4\cos^2\frac{\pi}{5} - 2\cos\frac{\pi}{5} - 1 = 0.$$

$\therefore \cos\dfrac{\pi}{5} = \dfrac{1+\sqrt{5}}{4}$, from solving the quadratic equation $4x^2 - 2x - 1 = 0$, taking the + sign because $\dfrac{\pi}{5}$ is acute.

Question 7

(a) (i) $\angle BMP = \angle BNP = 90°$

$\therefore BNPM$ is cyclic (opposite angles are supplementary)

(ii) $\angle BNM = \angle BPM$ (angles subtending the same arc are equal)

$\angle BPM = \angle BAP$ (alternate segment angles are equal)

$\therefore \angle BNM = \angle BAP$.

$\therefore NM \,//\, AP$ (corresponding angles are equal, then the lines are parallel).

(iii) In similar triangles TMN and TPA (equiangular, since $\angle BNM = \angle BAP$ and common angle T),

$\frac{r}{r+s} = \frac{p+q}{p+q+u}$ (corresponding sides in similar triangles are proportional)

$rp + rq + ru = rp + rq + sp + sq$

$ru = sp + sq = s(p+q)$

$\therefore \frac{s}{u} = \frac{r}{p+q}$

$\therefore \frac{s}{u} < \frac{r}{p}$.

(iv) $r < p$, since p is the hypotenuse, $\therefore \frac{s}{u} < 1, \therefore s < u$.

(b) (i) $\ddot{x} = R\omega^2$, where $\omega = \frac{2\pi}{T}. \therefore \ddot{x} = \frac{4\pi^2 R}{T^2}$.

At $x = R, \ddot{x} = -\frac{k}{R^2} = -\frac{4\pi^2 R}{T^2}, \therefore k = \frac{4\pi^2 R^3}{T^2}$.

(ii) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{4\pi^2 R^3}{T^2} \times \frac{1}{x^2}$

$\frac{1}{2}v^2 = \frac{4\pi^2 R^3}{T^2} \times \frac{1}{x} + C.$

When $x = R, v = 0, \therefore C = -\frac{4\pi^2 R^2}{T^2}$.

$\frac{1}{2}v^2 = \frac{4\pi^2 R^3}{T^2} \times \frac{1}{x} - \frac{4\pi^2 R^2}{T^2} = \frac{4\pi^2 R^2}{T^2}\left(\frac{R}{x} - 1\right).$

$\therefore v^2 = \frac{8\pi^2 R^2}{T^2}\left(\frac{R-x}{x}\right).$

(iii) $v = -\frac{\sqrt{8}\pi R}{T}\sqrt{\frac{R-x}{x}}$ (the minus sign is taken because the particle is moving towards the star).

$\frac{dx}{dt} = -\frac{\sqrt{8}\pi R}{T}\sqrt{\frac{R-x}{x}}.$

$-\int_R^0 \sqrt{\frac{x}{R-x}}\,dx = \frac{\sqrt{8}\pi R}{T}\int_0^{T_0} dt$, letting T_0 be the time taken to reach the star.

$\therefore \frac{\sqrt{8}\pi R}{T}T_0 = \left[R\sin^{-1}\sqrt{\frac{x}{R}} - \sqrt{x(R-x)}\right]_0^R = R\sin^{-1}(1) = \frac{\pi R}{2}.$

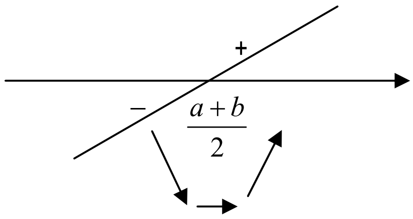
$\therefore T_n = \frac{T}{2\sqrt{8}} = \frac{T}{4\sqrt{2}}.$

Question 8

(a) (i) $f'(x) = \dfrac{3(abx)^{\frac{1}{3}} - (abx)^{-\frac{2}{3}}(ab)(a+b+x)}{9(abx)^{\frac{2}{3}}} = \dfrac{3abx - ab(a+b+x)}{9(abx)^{\frac{4}{3}}} = \dfrac{3x-(a+b+x)}{9(ab)^{\frac{1}{3}}x^{\frac{4}{3}}} = \dfrac{2x-(a+b)}{9(ab)^{\frac{1}{3}}x^{\frac{4}{3}}}$.

$f'(x) = 0$ gives $x = \dfrac{a+b}{2}$.

The sign of $f'(x)$, taking a, b, x all positive, is decided by the straight line $2x-(a+b)$ of positive gradient, so it is a minimum point.



(ii) When $x = \dfrac{a+b}{2}$, $f(x) = \dfrac{a+b+\frac{a+b}{2}}{3\left(ab\frac{a+b}{2}\right)^{\frac{1}{3}}} = \dfrac{\frac{3(a+b)}{2}}{3\frac{(ab)^{\frac{1}{3}}(a+b)^{\frac{1}{3}}}{2^{\frac{1}{3}}}} = \dfrac{(a+b)^{\frac{2}{3}}}{2^{\frac{2}{3}}(ab)^{\frac{1}{3}}}$.

$\therefore \dfrac{a+b+x}{3(abx)^{\frac{1}{3}}} \geq \dfrac{(a+b)^{\frac{2}{3}}}{2^{\frac{2}{3}}(ab)^{\frac{1}{3}}}$ (since this is the minimum point).

$\left(\dfrac{a+b+x}{3(abx)^{\frac{1}{3}}}\right)^3 \geq \left(\dfrac{(a+b)^{\frac{2}{3}}}{2^{\frac{2}{3}}(ab)^{\frac{1}{3}}}\right)^3 = \dfrac{(a+b)^2}{4ab} = \left(\dfrac{a+b}{2\sqrt{ab}}\right)^2$.

But $\dfrac{a+b}{2} \geq \sqrt{ab}$, so $\dfrac{a+b}{2\sqrt{ab}} \geq 1, \therefore \left(\dfrac{a+b+c}{3\sqrt[3]{abc}}\right)^3 \geq 1, \therefore \left(\dfrac{a+b+c}{3\sqrt[3]{abc}}\right) \geq 1, \therefore \dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$.

(iii) Let a, b, c be the roots then $a+b+c = p, abc = r, \therefore$ from (ii), $\dfrac{p}{3} \geq \sqrt[3]{r}. \therefore p^3 \geq 27r$.

(iv) $p^3 = 2^3 = 8 \ngeq 27r = 27$, so the equation $x^3 - 2x^2 + x - 1$ does not have three positive real roots.

$y' = 3x^2 - 4x + 1 = (3x-1)(x-1)$. For turning points, $y' = 0$ gives $x = \dfrac{1}{3}, 1$. As both turning points satisfy $x > 0$, while when $x = 0$, $y = -1$, so if it does not have three positive roots it must only have 1 positive root (and two complex conjugate roots).

(b) (i) $AP \times PB = (b\sec\theta - b\tan\theta)(b\sec\theta + b\tan\theta) = b^2(\sec^2\theta - \tan^2\theta) = b^2$.

(ii) $\angle OCD = \alpha - \beta$ (exterior angle is the sum of the two opposite interior angles)
$\angle CDB = 2\beta + (\alpha - \beta) = \alpha + \beta$ (exterior angle is the sum of the two opposite interior angles)

By the sine rule in ΔACP, $\dfrac{CP}{\sin(90° - \beta)} = \dfrac{AP}{\sin(\alpha-\beta)}, \therefore CP = \dfrac{AP\cos\beta}{\sin(\alpha-\beta)}$.

By the sine rule in ΔPDB, $\dfrac{PD}{\sin(90° - \beta)} = \dfrac{PB}{\sin(\alpha+\beta)}, \therefore PD = \dfrac{PB\cos\beta}{\sin(\alpha+\beta)}$.

(iii) $CP \times PD = \frac{AP\cos\beta}{\sin(\alpha-\beta)}\frac{PB\cos\beta}{\sin(\alpha+\beta)} = \frac{AP \times PB\cos^2\beta}{\sin(\alpha-\beta)\sin(\alpha+\beta)} = \frac{b^2\cos^2\beta}{\sin(\alpha-\beta)\sin(\alpha+\beta)}$, which depends on α and not θ.

(Note that β is constant)

(iv) From (iii), $CP \times PD = p(q+r)$ depends on α, but not θ.

Similarly, $QD.QC = q(p+r)$ depends on α, but not θ.

As α does not change for these collinear points, $CP \times PD = QD \times QC$.

$p(q+r) = q(p+r)$

$pq + pr = pq + qr$

$pr = qr$

$p = q$.

(v) The result in (iv) can be interpreted that $CP = DQ$, $\therefore$ the midpoint of CD is the same as the midpoint of PQ.
$\therefore$ As CD is moved to the position to be tangent to the hyperbola, the point of contact is also the midpoint of the chord UV.